Images and properties of a trigonometric function

I. Basic knowledge

1. Angle

The graph formed by rotating a radial line around its end point is called an angle. If rotate it by counterclockwise direction, then the angle is a positive angle; and if rotate it by clockwise direction, then the angle is a negative angle; if the rotation does not occur, then the angle is zero angle, The size of an angle can be any.

2. The angle measure

Divide a perigon into 360 equal parts, each equal part represents one degree. The radian measure: the central angle opposing to an arc equaling to the radius is called one radian, 360 degrees are equal to 2π radians. If the length of the arc opposing to a central angle is *L*, then the absolute value of its radians is $|\alpha|=\frac{L}{r}$, where r is the radius of the circle.

3. Trigonometric functions

In the Cartesian coordinate plane, place the vertex of the angle α at the origin point, and make the starting edge coincides with the positive half x axis, and take any point P different from the origin point, suppose the coordinates of P is (x,y) and the distance from which to the origin point is r, then $\sin\alpha=\frac{y}{r}$, $\cos\alpha=\frac{x}{r}$, $\tan\alpha=\frac{y}{x}$, $\cot\alpha=\frac{x}{y}$, $\sec\alpha=\frac{r}{x}$, $\csc\alpha=\frac{r}{y}$.

(1) The basic relational expressions of the trigonometric functions of the same angle

The reciprocal relations: $\tan\alpha=\frac{1}{\cot\alpha},\sin\alpha=\frac{1}{\csc\alpha},\cos\alpha=\frac{1}{\sec\alpha}$

The quotient relations: $\tan\alpha=\frac{\sin\alpha}{\cos\alpha},\cot\alpha=\frac{\cos\alpha}{\sin\varepsilon}$

The product relations: $\tan\alpha\times\cos\alpha=\sin\alpha,\cot\alpha\times\sin\alpha=\cos\alpha$

The square relations: $\sin^2\alpha+\cos^2\alpha=1, \tan^2\alpha+1=\sec^2\alpha, \cot^2\alpha+1=\csc^2\alpha$

(2) The induction formulas(for $k\frac{\pi}{2}+\alpha$, if k is an odd number, then sine, cosine, tangent, cotangent, secant and cosecant will transform into cosine, sine, cotangent，tangent. Cosecant and secant; if k is an even number, then the transformation does not occur. And the variation of the sign of corresponding function depends on the quadrant of $k\frac{\pi}{2}+\alpha$.)

① $\sin(\alpha+\pi)=-\sin\alpha, \cos(\alpha+\pi)=-\cos\alpha, \tan(\alpha+\pi)=\tan\alpha, \cot(\alpha+\pi)=\cot\alpha$

② $\sin(-\alpha)=-\sin\alpha, \cos(-\alpha)=\cos\alpha, \tan(-\alpha)=-\tan\alpha, \cot(-\alpha)=\cot\alpha$

③ $\sin(\pi-\alpha)=\sin\alpha, \cos(\pi-\alpha)=-\cos\alpha, \tan(\pi-\alpha)=-\tan\alpha, \cot(\pi-\alpha)=-\cot\alpha$

④ $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha, \cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha, \tan\left(\frac{\pi}{2}-\alpha\right)=\cot\alpha$

(3) The properties of $y=\sin x$（$x\in$R）

From the image of $y=\sin x$（$x\in$R）, we can have its following properties:

① The monotonous intervals: it is increasing over the interval $\left[2k\pi-\frac{\pi}{2}, 2k\pi+\frac{\pi}{2}\right]$ and decreasing over the interval $\left[2k\pi+\frac{\pi}{2}, 2k\pi+\frac{3}{2}\pi\right]$.

② Its least positive period is 2π.

③ It is an odd-function.

④ Boundedness: y takes its maximum 1 when and only when $x=2kx+\frac{\pi}{2}$, and y takes its minimum -1 when and only when $x=3k\pi-\frac{\pi}{2}$, where $k\in Z$. Its value domain is [-1, 1].

⑤ Symmetry: the lines $x=k\pi+\frac{\pi}{2}$ are all its symmetric axes, and the points $(k\pi,0)$ are all its symmetric centers.

(4) The properties of $y=\cos x$ ($x\in\mathbf{R}$)

From the image of $y=\cos x$ ($x\in\mathbf{R}$), we can have its following properties:

① The monotonous intervals: it is decreasing over the interval [$2k\pi$, $2k\pi+\pi$] and increasing over the interval [$2k\pi-\pi$, $2k\pi$].

② Its least positive period is 2π.

③ Parity: it is an even-function.

④ Symmetry: the lines $x=k\pi$ are all its symmetric axes, and the points $\left(k\pi+\frac{\pi}{2},0\right)$ are all its symmetric centers.

⑤ Boundedness: y takes its maximum 1 when and only when $x=2kx$, and y takes its minimum -1 when and only when $x=2k\pi-\pi$, where $k\in Z$. Its value domain is [-1, 1].

(5) The properties of $y=\tan x$($x\neq k\pi+\frac{\pi}{2}$)

Knowing from the image of $y=\tan x$($x\neq k\pi+\frac{\pi}{2}$),

① It is an odd-function;

② $y=\tan x$($x\neq k\pi+\frac{\pi}{2}$) is an increasing function over the open interval ($k\pi-\frac{\pi}{2}$, $k\pi+\frac{\pi}{2}$);

③ Its least positive period is π;

④ Its value domain is $(-\infty,+\infty)$;

⑤ The points $(k\pi, 0)$ and $(k\pi+\frac{\pi}{2}, 0)$ are all its symmetric center.

II. Typical examples

E.g.1: Let $x \in (0, \pi)$, compare the sizes of $\cos(\sin x)$ and $\sin(\cos x)$.

Solution: If $x \in \left[\frac{\pi}{2}, \pi\right)$, then $\cos x \le 1$ and $\cos x > -1$, so $\cos x \in \left(-\frac{\pi}{2}, 0\right]$,

so, $\sin(\cos x) \le 0$, and since $0 < \sin x \le 1$, thus $\cos(\sin x) > 0$, therefore, $\cos(\sin x) > \sin(\cos x)$.

If $x \in \left(0, -\frac{\pi}{2}\right]$, then since $\sin x + \cos x = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right) = \sqrt{2}\,(\sin x \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x) = \sqrt{2}$

$\sin(x+\frac{\pi}{4}) \leqslant \sqrt{2} < \frac{\pi}{2}$,

so, $0 < \sin x < \frac{\pi}{2} - \cos x < \frac{\pi}{2}$, thus $\cos(\sin x) > \cos(\frac{\pi}{2} - \cos x) = \sin(\cos x)$.

To sum up, $\cos(\sin x) < \sin(\cos x)$ is always true when $x \in (0, \pi)$.

E.g.2: Suppose α, β are acute angles, and $x \cdot (\alpha + \beta - \frac{\pi}{2}) > 0$. Show that $\left(\frac{\cos\alpha}{\sin\beta}\right)^x + \left(\frac{\cos\beta}{\sin\alpha}\right)^x < 2$.

Proof: If $\alpha + \beta > \frac{\pi}{2}$, then $x > 0$, from $\alpha > \frac{\pi}{2} - \beta > 0$, we have $\cos\alpha < \cos(\frac{\pi}{2} - \beta) = \sin\beta$, so, $0 < \frac{\cos\alpha}{\sin\beta} < 1$.

And since $\sin\alpha > \sin(\frac{\pi}{2} - \beta) = \cos\beta$, thus $0 < \frac{\cos\beta}{\sin\alpha} < 1$, therefore,

$$\left(\frac{\cos\alpha}{\sin\beta}\right)^x + \left(\frac{\cos\beta}{\sin\alpha}\right)^x < \left(\frac{\cos\alpha}{\sin\beta}\right)^0 + \left(\frac{\cos\beta}{\sin\alpha}\right)^0 = 2.$$

If $\alpha+\beta < \frac{\pi}{2}$, then $x<0$, from $0 <\alpha < \frac{\pi}{2}-\beta < \frac{\pi}{2}$, we have $\cos\alpha>\cos(\frac{\pi}{2}-\beta)=\sin\beta > 0$, so, $\frac{\cos\alpha}{\sin\beta}>1$.

And since $0<\sin\alpha<\sin(\frac{\pi}{2}-\beta)=\cos\beta$, thus $\frac{\cos\beta}{\sin\alpha}>1$, therefore,

$\left(\frac{\cos\alpha}{\sin\beta}\right)^x+\left(\frac{\cos\beta}{\sin\alpha}\right)^x<\left(\frac{\cos\alpha}{\sin\beta}\right)^0+\left(\frac{\cos\beta}{\sin\alpha}\right)^0=2$, the proof is done.

Notes: Above two examples all use the monotony and boundedness of trigonometric functions and the auxiliary angle formulas, the discussions of angels deserve to be noted.

E.g.3: Find the least positive period of $y=\sin(2\cos|x|)$.

Solution: First, $T=2\pi$ is a period of this function, in fact, since $\cos(-x)=\cos x$, so $\cos|x|=\cos x$. Second, since $|2\cos x|\leqslant 2<\pi$, so $y=0$ when and only when $x=k\pi+\frac{\pi}{2}$. So, if the least positive period is T_0, then $T_0=m\pi$, $m\in N_+$, and since $\sin(2\cos 0)=\sin 2\neq\sin(2\cos\pi)$, thus $T_0=2\pi$.

E.g.4: Find the maximum and the minimum of $y=\sin x+\sqrt{1+\cos^2 x}$.

Solution 1: Let $\sin x=\sqrt{2}\cos\theta, \sqrt{1+\cos^2 x}=\sqrt{2}\sin\theta\left(\frac{\pi}{4}\le 0\le\frac{3}{4}\pi\right)$, then we have $y=\sqrt{2}\cos\theta+\sqrt{2}\sin\theta=2\sin(\theta+\frac{\pi}{4})$. Since $\frac{\pi}{4}\le 0\le\frac{3}{4}\pi$, so $\frac{\pi}{2}\le\theta+\frac{\pi}{4}\le\pi$, thus $0\le\sin(\theta+\frac{\pi}{4})\leqslant 1$, therefore, $y_{min}=0$ when $\theta=\frac{3}{4}\pi$, i.e. $x=2k\pi-\frac{\pi}{2}$ $(k\in Z)$; $y_{max}=2$ when $\theta=\frac{\pi}{4}$, i.e. $x=2k\pi+\frac{\pi}{2}$ $(k\in Z)$.

Solution 2: Since $y=\sin x+\sqrt{1+\cos^2 x}\le\sqrt{2(\sin^2 x+1+\cos^2 x)}=2$ (due to $(a+b)^2\leqslant 2(a^2+b^2)$), and $|\sin x|\leqslant 1\leqslant\sqrt{1+\cos^2 x}$, so $0\leqslant\sin x+\sqrt{1+\cos^2 x}\leqslant 2$, thus $y_{max}=2$ when $\sqrt{1+\cos^2 x}=\sin x$, i.e. $x=2k\pi+\frac{\pi}{2}$ $(k\in Z)$; $y_{min}=0$ when $\sqrt{1+\cos^2 x}=-\sin x$, i.e. $x=2k\pi-\frac{\pi}{2}$ $(k\in Z)$.