Select valid symbols

Selecting valid symbols from the beginning in dealing with some problems tends to be the key for problem-solving.

For example, integers have various representing methods including the standard prime factor decomposition, the classification by modulus and various positional notations, etc. In solving a problem, we can select appropriate method representing numbers by the features of the problem so that the line of thinking become clear or the problem can be further researched.

E.g.1: Suppose positive integers p,q,r,a satisfy $pq=ra^2$, where r is a prime number, p,q prime with each other. Show that one of p,q is a complete square number.

Proof: suppose $p=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, q=q_1^{\beta_1}q_2^{\beta_2}\cdots q_l^{\beta_l}, a=a_1^{\gamma_1}a_2^{\gamma_2}\cdots a_m^{\gamma_m}$ are standard factor decompositions of p,q,a. then we have

$$p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_k^{\alpha_k}\cdot q_1^{\beta_1}\cdot q_2^{\beta_2}\cdots q_l^{\beta_l}=ra_1^{2\gamma_1}a_2^{2\gamma_2}\cdots a_m^{2\gamma_m}.$$

Since p,q prime with each other and r is a prime number, so each prime factor for the number among p,q that is not divisible by r has even powers, so this number must be a complete square number.

Notes: the basic arithmetic theorem means that any integer $n(n>1)$ has unique prime factor decomposition

$$n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k},$$

where $p_1<p_2<\cdots<p_k$ are different prime numbers, $\alpha_1, \alpha_2, \cdots, \alpha_k\in \mathbf{N}^*$ (sometimes we suppose $\alpha_1, \alpha_2, \cdots, \alpha_k\in \mathbf{N}$) for the convenience of considering problems).

This is an extremely important theorem, many problems like divisibility, co-prime, divisor and power can be considered once a positive integer is represented in the form of standard decomposition.In this problem, the standard decomposition of p,q,r is very convenient for solving problems by the properties of a prime number. Here are some basic results with the standard decomposition as the starting point.

(1) The number of positive divisors of above positive integer n is $d(n)=\prod_{i=1}^{k}(\alpha_i+1)$,

especially, the sufficient and necessary condition for n is a complete square number is that $d(n)$ is an odd number;

(2) The sum of all positive divisors of above positive integer n is $\sigma(n = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i-1}$;

(3) When $n \geq 2$, the number of positive integers that are less than n and co-prime with n is $\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)$.

E.g.2: (1) If n is a positive integer making $2n+1$ be a complete square number, show that: $n+1$ is the sum of two neighboring complete square numbers.

(2) If n is a positive integer making $3n+1$ be a complete square number, show that: $n+1$ is the sum of three complete square numbers.

Proof: (1) Since $2n+1$ is a complete square number, let $2n+1=a^2$, where a is an integer. Since a^2 is an odd number, so a is an odd number, thus, we can suppose $a=2k+1$, k is an integer, then

$$2n+1=(2n+1)^2,$$

so, $n = 2k^2 + 2k$,

thus, $n+1 = 2k^2+2k+1 = k^2+(k+1)^2$.

(2) Since $3n+1$ is a complete square number, let $3n+1=a^2$, a is an integer. Obviously, is not the multiples of three, so let $a=3k\pm1$, k is an integer, thus

$$3n+1=(3k\pm1)^2,$$

so, $n = 3k^2 \pm 2k$.

Therefore, $n+1 = 3k^2 \pm 2k + 1 = k^2+k^2+(k\pm1)^2$.

Notes: In the proof of (1), we classify a by its parity(i.e. the modulus is 2), then give its presentation($a=2k+1$). In the proof of (2), a is represented with $a=3k\pm1$ (i.e. the modulus is 3). So, the original problems are converted into simple algebraic problems, In solving the problems on integer, we tend to classify integers by residue class according the features of the problem, and discus separately each case, thus the problem is solved.

E.g.3: Whether the set $S=\{1,2,\cdots,3000\}$ has a subset consisted of 2000 elements?

Solution: Represent each positive integer with the form of $2^s t$, where s is a non-negative integer, t is an odd number.

If the set $A \subseteq S$ has the property: when $x \in A$, we have $2x \notin A$, then when $2^s t \in A$, we have $2^{s+1} t \notin A$. So, for each odd number, we have

$$|A \cap \{t, 2t, 2^2 t, \cdots\}| \leqslant |S \cap \{t, 2^2 t, 2^4 t, \cdots\}|,$$

where $|X|$ represents the number of elements in the finite set X. So, $|A|$ is no more than the number of elements in following set:

$$\{1, 3, \cdots, 2999, 1\times 2^2, 3\times 2^2, \cdots, 749\times 2^2, 1\times 2^4, 3\times 2^4, \cdots, 187\times 2^4, 1\times 2^6, 3\times 2^6, \cdots, 45\times 2^6, 1\times 2^8, 3\times 2^8, \cdots, 11\times 2^8, 1\times 2^{10}\}$$

that is,

$$|A| \leqslant 15\,000+375+94+23+6+1=1999<2000.$$

Thus, the set *A*having 2000 elements and satisfying the property stated in the problem does not exist.

Notes: the conditions in this problem involve the dependency relationships of *x* and 2*x* with the set *A*, so representing positive integers with the form of $2^s t$ (t is an odd number) contributes to the discussion, and the problem is solved.

E.g.4: For positive integer *m*, define*f*(*m*)as the number of the factor 2 in *m*! (i.e. the largest integer k atisfying $2^k | m!$). Show that: the number of *m* satisfying $m-f(m)=1000.$ is infinite.

Proof: Represent *m* in the binary form

$$m=\sum 2^{r_i}=2^{r_n}+2^{r_{n-1}}+\cdots+2^{r_1},$$

where $r_n > r_{n-1} > \cdots > r_1 \geqslant 0, r_i \in \mathbf{Z}.$ So,

$$f(m)=\left[\frac{m}{2}\right]+\left[\frac{m}{2^2}\right]+\left[\frac{m}{2^3}\right]+\cdots=\left[\frac{\sum 2^{r_i}}{2}\right]+\left[\frac{\sum 2^{r_i}}{2^2}\right]+\left[\frac{\sum 2^{r_i}}{2^3}\right]+\cdots$$
$$=\sum 2^{r_i-1}+\sum 2^{r_i-2}+\sum 2^{r_i-3}+\cdots,$$

where summing signs only operate on the items with non-negative exponents. Further, we have

$$f(m)=\sum(2^{r_i-1}+2^{r_i-2}+\cdots+1)=\sum(2^{r_i}-1)=m-n.$$

So, $m-f(m)=n$, that is, $m-f(m)$ is equal to the number of non-zero numbers of m in the binary form.

Since there are infinite positive integers *m* so that exactly 1000 non-zero numbers exist in their binary representations, so the proposition is proved.

Notes: the definition of *f*(*m*) indicates us to represent *m*in binary form, which is also convenient for describing the properties of *m*satisfying the conditions. A generalized conclusion can be obtained with the method used in this problem.

For any positive integer *m*, define $f_p(m)$ as the number of the prime factor p in *m*!(i.e. the largest integer k satisfying $p^k \mid m!$), then the sum of digits of *m* represented in p-scale is equal to $\frac{m-f_p(m)}{p-1}$.

Exercises

1.Suppose *f*(*n*) is a function from N* to N*, *f*(*n*)=1, and for any $n\in\mathbf{N}^*$, $\varepsilon\in\{0, 1\}$, we have $f(2n+\varepsilon)=3f(n)+\varepsilon$. Find the value domain of *f*(*n*).

2. Find the number of non-empty subsets of $S=\{1, 2, \cdots, n\}$ containing no two neighboring integers.

3. Show that: the sum of the squares of four edges of any quadrilateral is equal to the sum of the squares of two diagonals together with four times of the square of the line segment connecting middle points of two diagonals.

4. The tangents at A and B of the circumcircle of the acute triangle ABC intersect at D, and M is the middle point of AB. Show that: $\angle ACM=\angle BCD$.(The training problem for the 2007 national team)

Answers for the exercises

1. ***Solution:*** Calculate first some concrete values of $f(n)$:

n	$(1)_2$	$(10)_2$	$(11)_2$	$(100)_2$	$(101)_2$	$(110)_2$	$\cdots$
$f(n)$	$(1)_3$	$(10)_3$	$(11)_3$	$(100)_3$	$(101)_3$	$(110)_3$	$\cdots$

So, we guess that $f(n)=(\overline{a_k a_{k-1}\cdots a_1})_3$ is established for $n=(\overline{a_k a_{k-1}\cdots a_1})_2$.

Next, we use the mathematical induction to prove above conclusion for the digits k of $n=(\overline{a_k a_{k-1}\cdots a_1})_2$.

Obviously, the conclusion is true when $k=1$.

Suppose above conclusion is true when n is a number of k digits, and consider any number of $k+1$ digits, $n_1 = (\overline{a_k a_{k-1}\cdots a_0})_2$.

In $f(2n+\varepsilon)=3f(n)+\varepsilon$, let $n=(\overline{a_k a_{k-1}\cdots a_1})_2$, $\varepsilon=a_0\in\{0,1\}$, and since

$$2n+\varepsilon=2\cdot(\overline{a_k a_{k-1}\cdots a_1})_2+a_0=(\overline{a_k a_{k-1}\cdots a_1 a_0})_2=n_1, \quad \text{so}$$

$$f(n_1)=3f(n)+a_0=3\cdot(\overline{a_k a_{k-1}\cdots a_1})_3+a_0=(\overline{a_k a_{k-1}\cdots a_1 a_0})_3,$$

thus, above conclusion is also true under the case of $k+1$. Knowing from the mathematical induction, above conclusion is true.

So, the value domain of $f(n)$is the set of all positive integers containing only digits 0 and 1 in their ternary representations, that is

$$\{3^{r_1}+3^{r_2}+\cdots+3^{r_s}\mid s\in\mathbf{N}^*,\ r_1,r_2,\cdots,r_s\in\mathbf{N},\ r_1>r_2>\cdots>r_s\}.$$

Note 1: This problem explores first, calculating some concrete values of $f(n)$. Once substitute the decimal representations of n and $f(n)$with appropriate positional notations(the binary notation and the ternary notation), the valuing rule will be presented under the new notation, which is convenient for the writing. The subsequent proof is very easy.

Note 2: The conclusion in this problem is exactly used in a certain problem of CMO'95.

(CMO'95) The function $f: \mathbf{N}^* \to \mathbf{N}^*$ satisfies $f(1)=1$, and for any $n \in \mathbf{N}^*$, we have

$$3f(n)f(2n+1) = f(2n)(1+3f(n)),\ f(2n) < 6f(n).$$

Find all solutions for $f(k)+f(l)=293,\ k<l$.

Next, we present another representation method of positive integers, the Fibonacci representation of positive integers. We can define $\{F_n\}$ as follows:

$$F_1 = 1,\ F_2 = 2,\ F_{n+2} = F_{n+1} + F_n.$$

We consider representing a positive integer as the sum of some different items in $\{F_n\}$. For example,

$$10 = 8 + 2 = F_5 + F_2,$$

$$30 = 13 + 8 + 5 + 3 + 1 = F_6 + F_5 + F_4 + F_3 + F_1,$$

but 30 can also be represented as

$$30 = 21 + 8 + 1 = F_7 + F_5 + F_1.$$

If further requirement presenting a positive integer as the sum of some items of $\{F_n\}$ that have no two adjacent items, then among above two representations of the positive integer 30, only the later satisfies the requirement.

Generally, the following important conclusion can be proved with the mathematical induction.

Theorem: Each positive integer can be uniquely represented as the sum of some items of $\{F_n\}$ that have no two adjacent items.

Now, if $n = a_kF_k + a_{k-1}F_{k-1} + \cdots + a_1F_1$, where $a_1, a_2, \cdots, a_k \in \{0, 1\}$ and two neighboring items do not take 1 simultaneously, we denote n with $(\overline{a_ka_{k-1}\cdots a_1})_F$, which is called the Fibonacci representation.

Sometimes, using this representation method can solve some problems succinctly.

2. ***Solution:*** Knowing from the properties of the Fibonacci representation of a positive integer, each

non-empty subset A of S containing no two neighboring integers exactly corresponds to a positive integer $m = (\overline{a_n a_{n-1} \cdots a_1})_F$ being less than $(\overline{1\underbrace{00\cdots0}_{n个0}})_F = F_{n+1}$, where it is only needed to specify

$$a_i = \begin{cases} 1, & i \in A, \\ 0, & i \notin A. \end{cases}$$

Thus, the number of non-empty subsets satisfying the conditions is $F_{n+1} - 1$, that is, $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right] - 1$

3. ***Proof:*** Suppose the coordinates in the Cartesian coordinate system of the four vertexes of the quadrilateral are respectively $A_i(x_i, y_i)$, $i = 1, 2, 3, 4$, then the coordinates of the middle points M, N of A_1A_3, A_2A_4 are respectively $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$, $\left(\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}\right)$.

Since,

$$\begin{aligned} & 4\left(\frac{x_1+x_3}{2} - \frac{x_2+x_4}{2}\right)^2 + (x_1-x_3)^2 + (x_2-x_4)^2 \\ &= (x_1+x_3-x_2-x_4)^2 + (x_1-x_3)^2 + (x_2-x_4)^2 \\ &= 2(x_1^2+x_2^2+x_3^2+x_4^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_1) \\ &= (x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_4)^2 + (x_4-x_1)^2, \end{aligned}$$

Similarly, we have

$$\begin{aligned} & 4\left(\frac{y_1+y_3}{2} - \frac{y_2+y_4}{2}\right)^2 + (y_1-y_3)^2 + (y_2-y_4)^2 \\ &= (y_1-y_2)^2 + (y_2-y_3)^2 + (y_3-y_4)^2 + (y_4-y_1)^2, \end{aligned}$$

Adding above two expressions, and according to the distance formula of two points, we have

$$\begin{aligned} & 4|MN|^2 + |A_1A_3|^2 + |A_2A_4|^2 \\ &= |A_1A_2|^2 + |A_2A_3|^2 + |A_3A_4|^2 + |A_4A_1|^2, \end{aligned}$$

So, the proposition is established.

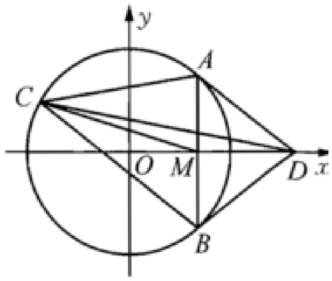
Notes: The analytical method is used for solving this problem. Generally, solving a problem with the analytical method has some following rules.

If a right angle occurs in a geometric problem, then we can consider making the coordinate axles be two edges of the right angle; using the analytical geometry tends to be relatively convenient for the problems involving the square relationships(this problem is such a good example); for some problems, replacing letters with numbers brings no effects on the nature of the problem, conversely, numbers can also be replaced with letters sometimes, which will make the internal structure obvious.

Surely, how to select more appropriate notation depends on the concrete problem. In this problem, we select the most general coordinates for performing algebraic operations, which is convenient for keeping the symmetry of an expression and proving while observing, and since the symmetric relationships between the horizontal coordinates and the vertical coordinates, we only need to prove the equations expressed with the horizontal coordinates, then the equations expressed with the vertical coordinates will be similarly obtained, which produces the effects of doing more with less.

4. ***Proof:*** Use the method of complex number.

Suppose the circumcircle of $\triangle ABC$ is the unit circle on the complex plane with O as its circle center, and the direction of the radiant line OM represents the positive direction of the real axis.



Knowing from the known condition that D is on the prolonged line of OM, and $OM \cdot OD = OA^2 = 1$.

Suppose A, B respectively corresponds to the complex numbers z, $\bar{z}$, then M corresponds to the complex number $\operatorname{Re} z$, corresponds to the complex number $\frac{1}{\operatorname{Re} z}$, and suppose C corresponds to the complex number *c*.

Obviously, $\angle ACM$ and $\angle BCD$ are all acute angles, so we only need to prove

$$H = \frac{c - z}{c - \frac{1}{\operatorname{Re} z}} : \frac{c - \operatorname{Re} z}{c - \bar{z}} \in \mathbf{R}$$, i.e. $\overline{H} = H$.

Let $(c - z)(c - \bar{z}) = P$, $\left(c - \frac{1}{\operatorname{Re} z}\right)(c - \operatorname{Re} z) = Q$, then $H = \frac{P}{Q}$.

Observing that $z \cdot \bar{z} = c \cdot \bar{c} = 1$, then

$$\overline{P} = (\overline{c} - \overline{z})(\overline{c} - z) = \left(\frac{1}{c} - \frac{1}{z}\right)\left(\frac{1}{c} - \frac{1}{\overline{z}}\right) = \frac{1}{c^2}(z - c)(\overline{z} - c) = \frac{P}{c^2},$$

$$\overline{Q} = \left(\frac{1}{c} - \frac{1}{\operatorname{Re} z}\right)\left(\frac{1}{c} - \operatorname{Re} z\right) = \frac{1}{c^2 \operatorname{Re} z}(\operatorname{Re} z - c)(1 - c\operatorname{Re} z)$$
$$= \frac{1}{c^2 \operatorname{Re} z} \cdot Q\operatorname{Re} z = \frac{Q}{c^2},$$

so, $\overline{H} = \frac{\overline{P}}{\overline{Q}} = \frac{P}{Q} = H$. Thus $\angle ACM = \angle BCD$.

Notes: In this problem, the positions of A, B are symmetric, while D, M closely associate with A, B. So, without losing generality, respectively correspond A, B to the complex numbers z, $\bar{z}$ on the unit circle, and convert the conclusion of angular equality that needs to be proved into such an expressions that a complex number expression takes real number.

The method of complex number is an algebraic method, however, the multiple and division of complex numbers have clear geometric meaning in such aspects as modulus and complex argument, which is its merits. Some representation forms of a complex number, such as the algebraic form, the trigonometric form and the exponential form, can also be selected appropriately in solving a problem, which reveals the associations between algebraic, trigonometric and geometric knowledge. any problem related with rotation and homothety can usually be solved with the method of complex number.