Solving a triangle

I. Basic knowledge

1. The sine theorem: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$ (R represents the radius of its circumcircle)

Proposition 1: the area of ΔABC is $S_{\Delta ABC}=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B.$

Proposition 2: for ΔABC, we have $b\cos C+c\cos B=a$.

Proposition 3: for ΔABC, $A+B=\theta$, the solution α satisfies $\frac{a}{\sin\alpha}=\frac{b}{\sin(\theta-\alpha)}$, then $\alpha=A$.

2. The cosine theorem: $c^2=a^2+b^2-2ab\cos C$, $\cos C=\frac{a^2+b^2+c^2}{2ab}$.

3. The projection theorem: $a=b\cos C+c\cos B.$

4. The formula calculating the area of a triangle:

$S_{\triangle ABC}=\frac{1}{2}ab\sin C=p\cdot r,$ where $p=\frac{1}{2}(a+b+c)$, r is the radius of its incircle.

$S_{\triangle ABC}=\sqrt{p(p-a)(p-b)(p-c)}$

5. The Stewart's theorem:

For ΔABC, D is any point on BC, $BD=p, DC=q$, then $AD^2=\frac{b^2p+c^2q}{p+q}-pq.$

II. Typical examples

E.g.1: Suppose three interior angles A,B,C of ΔABC form an arithmetic progression, and $\frac{1}{\cos A}+\frac{1}{\cos C}=-\frac{\sqrt{2}}{\cos B}$, then find the value of $\cos\frac{A-C}{2}$.

Solution: Since $A=120^0$-C, so $\cos\frac{A-C}{2}$=cos(60^0-C), And because $\frac{1}{\cos A}+\frac{1}{\cos C}=\frac{1}{\cos(120^0-C)}+\frac{1}{\cos C}=\frac{\cos(120^0-C)+\cos C}{\cos C\cos(120^0-C)}$

$=\frac{2\cos 60^0\cos(60^0-C)}{\frac{1}{2}[\cos 120^0+\cos(120^0-2C)]}=\frac{2\cos(60^0-C)}{\cos(120^0-2C)-\frac{1}{2}}=-2\sqrt{2}$,

So, $4\sqrt{2}\cos^2\frac{A-C}{2}+2\cos\frac{A-C}{2}-3\sqrt{2}=0$.

Then we have $\cos\frac{A-C}{2}=\frac{\sqrt{2}}{2}$ 或 $\cos\frac{A-C}{2}=-\frac{3\sqrt{2}}{8}$.

And $\cos\frac{A-C}{2}>0$, thus $\cos\frac{A-C}{2}=\frac{\sqrt{2}}{2}$.

E.g.2: As shown in the figure, M and N are respectively the middle points of the arcs $\overset{\frown}{BC}$ and $\overset{\frown}{AC}$, which locate at the circumcircle Γ of the acute triangle ΔABC ($\angle A<\angle B$). Draw a line PC so that $PC\,//\,MN$, which intersects with Γ at P, and I is the incenter of ΔABC, connecting PI and extending it to intersect with Γ at T.

(1) Show that $MP\cdot MT=NP\cdot NT$.

(2) Take any point Q on the arc $\overset{\frown}{AB}$ without C ($Q\neq A$, T, B), and respectively denote the incenters of ΔAQC and $\triangle QCB$ as I_1 and I_2, show that: four points Q, I_1, I_2 and T are concyclic.

Analysis: (1) connect NI and MI, since $PC\,//\,MN$, and P, C, M, N are concyclic, so $NP=MI$, $PM=NI$. So, the quadrilateral $MPNI$ is a parallelogram. Thus, $S_{\triangle PMT}=S_{\triangle PNT}$ (same bottom edge and same height).

Since P, N, T, M are concyclic, so $\angle TNP+\angle PMT=180°$, and from the area formula of a triangle, we have $PM\cdot MT=PN\cdot NT$.

(2) Since $\angle NCI_1=\angle NCA+\angle ACI_1=\angle NQC+\angle QCI_1=\angle CI_1N$,

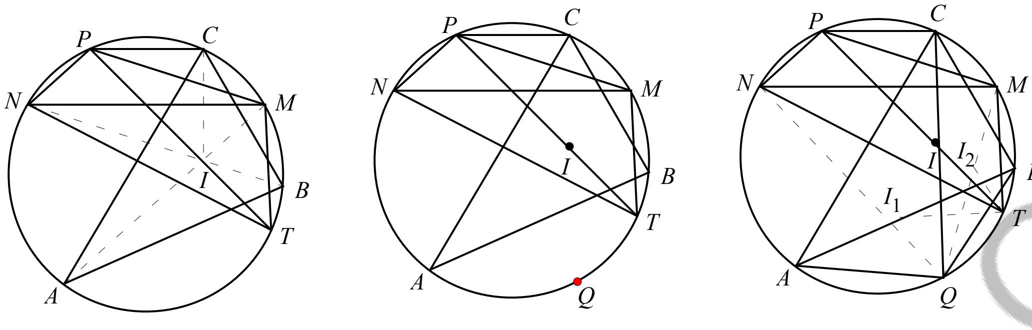
so, $NC=NI_1$, similarly, $MC=MI_2$. From $MP\cdot MT=NP\cdot NT$, we have $\frac{NT}{MP}=\frac{MT}{NP}$.

From (1), $MP=NC$, $NP=MC$, so $\frac{NT}{NI_1}=\frac{MT}{MI_2}$.

And since $\angle I_1NT=\angle QNT=\angle QMT=\angle I_2MT$, we have $\Delta I_1NT\backsim\Delta I_2MT$.

So $\angle NTI_1 = \angle MTI_2$, thus $\angle I_1QI_2 = \angle NQM = \angle NTM = \angle I_1TI_2$.

Therefore, Q, I_1, I_2 and T are concyclic.



E.g.3: Three edges a,b,c of ΔABC satisfy $a+b\geq 2c$, and A,B,C are the interior angles of ΔABC. Show that: $C \leq 60^0$.

Solution 1: From the sine theorem, $a+b\geq 2c \Leftrightarrow \sin A+\sin B \geq 2\sin C \Leftrightarrow 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \geq$ $2\cdot 2\sin\frac{C}{2}\cos\frac{C}{2} \Leftrightarrow \cos\frac{A-B}{2} \geq 2\sin\frac{C}{2}$, and since $\cos\frac{A-B}{2} \leq 1$, so $2\sin\frac{C}{2} \leq 1$. Observing that $0<\frac{C}{2}<90^0$, so $\frac{C}{2} \leq 30^0$, thus $C \leq 60^0$.

Solution 2: From the cosine theorem, $\cos C = \frac{a^2+b^2-c^2}{2ab} \geq \frac{a^2+b^2-\left(\frac{a+b}{2}\right)^2}{2ab}$ （since $c \leq \frac{a+b}{2}$）

$= \frac{4(a^2+b^2)-(a+b)^2}{8ab} = \frac{3(a^2+b^2)-2ab}{8ab} = \frac{3(a^2+b^2)}{8ab} - \frac{1}{4} \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$, so $C \leq 60^0$.

E.g.4: A,B,C are the interior angles of ΔABC, and ΔABC is not a right-angled triangle.

(1) Show that: $\tan A+\tan B+\tan C = \tan A\tan B\tan C$;

(2) When $\sqrt{3}\tan C - 1 = \frac{\tan B+\tan C}{\tan A}$, and the reciprocals of $\sin 2A, \sin 2B, \sin 2C$ form an arithmetic progression, find the value of $\cos\frac{C-A}{2}$.

(1) ***Proof:*** $A+B+C=\pi$, $A+B=\pi-C$, taking tangent on both sides, we have $\tan(A+B)=\tan(\pi-C)$, $\frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C \Rightarrow \tan A+\tan B+\tan C = \tan A\tan B\tan C$.

(2) ***Solution:*** $\sqrt{3}\tan A\tan C - \tan A = \tan B+\tan C, \sqrt{3}\tan A\tan C = \tan A+\tan B+\tan C$.

Knowing from (1), $\sqrt{3}\tan A\tan C=\tan A\tan B\tan C$, so $\tan B=\sqrt{3}, B=\frac{\pi}{3}$. And since

$\frac{2}{\sin 2B}=\frac{1}{\sin 2A}+\frac{1}{\sin 2C}$, so $\frac{\sin 2A+\sin 2C}{\sin 2A\sin 2C}=\frac{2}{\sin 2B}=\frac{2}{\frac{\sqrt{3}}{2}}=\frac{4}{\sqrt{3}}$. That is,

$\frac{2\sin(A+C)\cos(A-C)}{-\frac{1}{2}[\cos(2A+2C)-\cos(2A-2C)]}=\frac{4}{\sqrt{3}}$.Substituting $A+C=\frac{2}{3}\pi$ into above expression,

$\frac{\sqrt{3}\cos(A-C)}{-\frac{1}{2}\left[-\frac{1}{2}-\cos(2A-2C)\right]}=\frac{4}{\sqrt{3}}, 3\cos(A-C)=1+2\cos(2A-2C)=4\cos^2(A-C)-1$.

$4\cos^2(A-C)-3\cos(A-C)-1=0, \cos(A-C)=1$ (ΔABC is now an equilateral triangle) or $-\frac{1}{4}$.Since

$\cos\frac{C-A}{2}>0$, so $\cos\frac{C-A}{2}=\sqrt{\frac{1+\cos(C-A)}{2}}=1$ or $\frac{\sqrt{6}}{4}$.