The Gaussian function

Basic knowledge

1. Definition

Let $x \in R$, use $[x]$ to denote the largest integer no more than x. We call $y=[x]$ the Gaussian function, also rounding function. Obviously, the definition domain of $y=[x]$ is R, and the value domain is Z .Any real number can be written as the sum of the integer fraction and the non-negative pure decimal fraction, i.e. $x=[x]+a(0 \le a<1)$, so $<[x]+1$, where $[x]$ is the integer fraction of x, and $\{x\}=x-[x]$ is the decimal fraction.

2. Properties

(1) $y=[x]$ is a non-decreasing and unbounded function expressed in segments, that is, when $x_1 \le x_2$, $[x_1] \le [x_2]$ is established ;

(2) $[n+x]=n+[x]$, where $n \in Z$;

(3) $x-1<[x] \le x<[x]+1$;

(4) If $[x]=[y]=n$, then $x=n+a, y=n+b,$, where $0 \le a,b<1$;

(5) For all real numbers x,y, $[x]+[y] \le [x+y]$ is established;

(6) If $x \ge 0, y \ge 0$, then $[xy] \ge [x][y]$;

(7) $$[-x]=\begin{cases} -[x]-1 & (x \text{ is not an integer}) \\ -[x] & (x \text{ is an integer}) \end{cases}$$

(8) If $n \in N^+$, then $\left[\frac{[x]}{n}\right]=\left[\frac{x}{n}\right]$; when $n=1$, $[[x]]=[x]$;

(9) If integers a,b satisfy $a=bq+r$ ($b>0, q, r$ is an integer, $0 \le r<b$), then $\left[\frac{a}{b}\right]=q$;

(10) x is a positive real number, n is a positive integer, then there are totally $\left[\frac{x}{n}\right]$ multiples of n among the positive integers no more than x.

(11) Suppose p is any prime number, and denote the highest power containing p in $n!$ as $p(n!)$, then we have

$$p(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^m}\right]\left(p^m \le n < p^{m+1}\right).$$

Proof: Since p is a prime number, all powers containing p in $n!$ are equal to the sum of the power of each factor $1,2,\cdots,n$ containing p in $n!$. Knowing from the property 10, there are $\left[\frac{n}{p}\right]$ multiples of p, $\left[\frac{n}{p^2}\right]$ multiples of p^2, $\left[\frac{n}{p^3}\right]$ multiples of p^3,..., when $p^m \le n < p^{m+1}$, $\left[\frac{n}{p^{m+1}}\right]=\left[\frac{n}{p^{m+2}}\right]=\cdots=0$, so the proposition is established.

The Gaussian function is an important mathematical concept, which has continuous definition domain and discrete value domain. Since the Gaussian function associates with both the continuity and discreteness, so it has unique properties and extensive applications.

Solving the problems on the Gaussian function needs many kinds of mathematical thinking ways, where the classification discussion(e.g. classify a interval), the proposition transformation, the method combining numbers with shapes, the method of completing a integer, the estimation,etc. are commonly observed.

Typical examples

E.g.1: If a real number r satisfies $\left[r+\frac{19}{100}\right]+\left[r+\frac{20}{100}\right]+\cdots+\left[r+\frac{91}{100}\right]=546$, find $[100r]$.

Solution: There are 73 items on the left side of the equation, and since $\frac{19}{100},\frac{20}{100},\cdots,\frac{91}{100}$ are all less than 1, so each item is $[r]$ or $[r]+1$. Observing that $73\times7<546<73\times8$, thus there must be $[r]=7$. Further, we have $73\times7+35=546$, so the items from the first item to the thirty-eighth item in the left side of the original equation are all 7, and each item starting from the thirty-ninth item is

8, that is, $\left[r+\frac{56}{100}\right]=7;\left[r+\frac{57}{100}\right]=8.\therefore r+0.56<8, r+0.57\geq 8\therefore 7.43\leq r<7.44$

E.g.2: Find the value of $\sum_{n=1}^{100}\left[\frac{23n}{101}\right]$.

Solution: Knowing from the problem, for any $n\in\{1,2,\cdots,100\},\frac{23n}{101}\notin Z$, since $\frac{23n}{101}+\frac{23(101-n)}{101}=23$, so $\left\{\frac{23n}{101}\right\}+\left\{\frac{23(101-n)}{101}\right\}=1;\left[\frac{23n}{101}\right]+\left[\frac{23(101-n)}{101}\right]=22.$ Thus, $\sum_{n=1}^{100}\left[\frac{23n}{101}\right]=22\times 50=1100$.

Notes: This example adopts the thought of completing an integer by group.

E.g.3: For a natural number n and all real numbers x, show that:

$[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right]=[nx]$. (The Hermite Equation)

Proof: For any natural number n, construct the function

$f(x)=[nx]-[x]-\left[x+\frac{1}{n}\right]-\left[x+\frac{2}{n}\right]-\cdots-\left[x+\frac{n-1}{n}\right]$, then

$f\left(x+\frac{1}{n}\right)=[nx]+1-\left[x+\frac{1}{n}\right]-\left[x+\frac{2}{n}\right]-\cdots-\left[x+\frac{n-1}{n}\right]-([x]+1)=f(x)$, so, $f(x)$ is a periodic function with $T=\frac{1}{n}$ as its period, therefore, the original proposition only needs to prove $f(x)=0$ is established over $\left[0,\frac{1}{n}\right]$. And obviously, this conclusion is established.

E.g.4: For any $n\in N^{+}$, show that

$\left[\sqrt{n}+\sqrt{n+1}\right]=\left[\sqrt{4n+1}\right]=\left[\sqrt{4n+2}\right]=\left[\sqrt{4n+3}\right]$.

Proof: Prove first $\left[\sqrt{4n+1}\right]+1>\sqrt{4n+3}$. Let $x=\left[\sqrt{4n+1}\right]+1$, then $x^2>4n+1$.

When $x=2m\left(m\in Z^{+}\right)$, $x^{2}=4m^{2}>4n+1$, so $m^{2}\geq n+1$, thus $x^{2}=4m^{2}\geq 4n+4>4n+3$;

When $x=2m-1\left(m\in Z^{+}\right)$, $x^{2}=4m^{2}-4m+1>4n+1$, $m^{2}-m>n$, i.e. $m^{2}-m\geq n+1$, thus

$x^{2}=4\left(m^{2}-m\right)+1\geq 4n+5>4n+3$. Therefore, the proposition is established, that is

$\left[\sqrt{4n+1}\right]\leq\sqrt{4n+1}<\sqrt{4n+2}<\sqrt{4n+3}<\left[\sqrt{4n+1}\right]+1$, so $\left[\sqrt{4n+1}\right]=\left[\sqrt{4n+2}\right]=\left[\sqrt{4n+3}\right]$.

And since $\left(\sqrt{n}+\sqrt{n+1}\right)^{2}=2n+1+2\sqrt{n^{2}+n}>2n+1+2n=4n+1$

$\left(\sqrt{n}+\sqrt{n+1}\right)^{2}=2n+1+2\sqrt{n^{2}+n}<2n+1+2\left(n+1\right)=4n+3$

So, $\sqrt{4n+1}<\sqrt{n}+\sqrt{n+1}<\sqrt{4n+3}$

$\left[\sqrt{n}+\sqrt{n+1}\right]=\left[\sqrt{4n+1}\right]=\left[\sqrt{4n+2}\right]=\left[\sqrt{4n+3}\right]$

Notes: The proof of this example adopts the squeeze theorem.

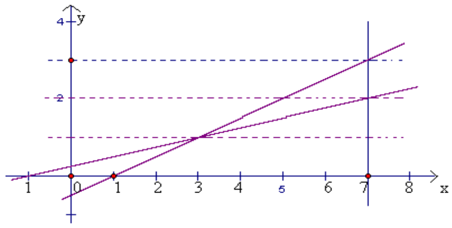
E.g.5: Solve the equation $\left[\frac{5+6x}{8}\right]=\frac{15x-7}{5}$.

Solution: Let $\frac{15x-7}{5}=n\left(n\in Z\right)$, then $x=\frac{5n+7}{15}$, substituting it into the original equation and tidying the equation, we have $\left[\frac{10n+39}{40}\right]=n$, knowing from the definition of the Gaussian function, we have $0\leq\frac{10n+39}{40}-n<1$, so $-\frac{1}{30}<n\leq\frac{13}{10}$, thus $n=0,n=1$. If $n=0$, then $x=\frac{7}{15}$; If $n=1$, then $x=\frac{4}{5}$.

Notes: The equation in this example is of the type $\left[u\right]=v$, which is typically solved by the definition and properties of the Gaussian function together with the U-substitution method.

E.g.6: Solve the equation $\left[\frac{x+1}{4}\right]=\left[\frac{x-1}{2}\right]$.

Solution: From the properties of the Gaussian function, we have $-1<\frac{x+1}{4}-\frac{x-1}{2}<1$, that is $-1<x<7$. Let $y_1=\frac{x+1}{4}, y_1=\frac{x-1}{2}$, and draw the images of these two in the same coordinate system. Analyze the images of these two over $(-1,7)$, obviously, $\left[\frac{x+1}{4}\right]=0$ when $x\in(-1,1)$,



and $\left[\frac{x-1}{2}\right]=-1$, so the equation is not established; $\left[\frac{x+1}{4}\right]=\left[\frac{x-1}{2}\right]=0$ when $x\in[1,3)$; $\left[\frac{x+1}{4}\right]=\left[\frac{x-1}{2}\right]=1$ when $x\in[3,5)$; $\left[\frac{x+1}{4}\right]=1$ when $x\in[5,7)$, and $\left[\frac{x-1}{2}\right]=2$, so the equation is not established. To sum up, the solution for the original equation is $\{x|1\le x<5\}$.

Notes: The equation in this example is of the type $[u]=[v]$. First, solve the valuing range of x from $-1<u-v<1$. However, this condition is the sufficient condition making the original equation be established, but not the necessary condition, so we still need to analyze the images of $u=f(x)$ and $v=g(x)$ for obtaining the correct results.

E.g.7: Solve the equation $3x^3-[x]=3$.

Solution: Discussing by intervals is an effective method for the equations with higher powers containing $[x]$.

If $x\le -1$, then $3x^3-[x]\le 3x-x+1=2x+1<0$, the original equation is established;

If $-1<x\le 0$, then $3x^3-[x]=3x^3-(-1)=3x^3+1\le 1$, the original equation is not established;

If $0\le x<1$, then $3x^3-[x]=3x^3-0=3x^3<3$, the original equation is not established;

If $1\le x<2$, then $3x^3-[x]=3x^3-1$, the original equation is $3x^3=4$, so we have $x=\sqrt[3]{\frac{4}{3}}$;

If $x\ge 2$, then $3x^3-[x]\ge 3x^3-x>3x-x=2x>4$, the original equation is not established;

So, the solution for the original equation is $\left\{x \middle| x=\sqrt[3]{\frac{4}{3}}\right\}$.

E.g.8: Show that: if p is a prime number larger than 2, then $\left[\left(2+\sqrt{5}\right)^p\right]-2^{p+1}$ can be divisible by p.

Proof: This example adopts the construction method.

Knowing from the binomial theorem, for any $p \in Z$, $\left(2+\sqrt{5}\right)^p+\left(2-\sqrt{5}\right)^p$ is an integer, and since $-1<\left(2-\sqrt{5}\right)^p<1, \therefore \left(2+\sqrt{5}\right)^p+\left(2-\sqrt{5}\right)^p=\left[\left(2+\sqrt{5}\right)^p\right]$, so we have $\left[\left(2+\sqrt{5}\right)^p\right]-2^{p+1}=2\left(C_p^2 \cdot 2^{p-2} \cdot 5+C_p^4 \cdot 2^{p-4} \cdot 5^2+\cdots+C_p^{p-1} \cdot 2 \cdot 5^{\frac{p-1}{2}}\right)$, where p is a prime number.

Since $C_p^k=\frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}(k=2,4,\cdots,p-1)$ are all divisible by the prime number p, thus the original proposition is established.