The extreme values of a function

1. The definitions of the extreme values

Generally, the extreme values of a function can be classified into the minimum and the maximum: suppose the definition domain of $y = f(x)$ is T, $x_0 \in T$, and the function value at x_0 is $f(x_0)$.

If the inequality $f(x) \geq f(x_0)$ is always true for any x in T, then call $f(x_0)$ the minimum of $y = f(x)$, and denote it $y_{\min} = f(x_0)$;

If the inequality $f(x) \leq f(x_0)$ is always true for any x in T, then call $f(x_0)$ the maximum of $y = f(x)$, and denote it $y_{\max} = f(x_0)$.

The extreme values of a function typically have two special cases.

(1) If $f(x)$ increases or decreases monotonously over $[a,b]$, then $f(a)$ is the minimum or maximum of $f(x)$ over $[a,b]$, $f(b)$ is the maximum or minimum of $f(x)$ over $[a,b]$.

(2) If a continuous function $f(x)$ has only one extremely largest or smallest value over (a,b), and has no extremely smallest or largest value , then the extremely largest or smallest value is the maximum or minimum of $f(x)$ over (a,b).

2. Exploring on the methods solving the extreme values of a function

Mathematical knowledge on the extreme values in the middle school provide the basis for further learning the problems on the extreme values in the higher mathematics, so the problems on the extreme values have been the hot-topics in various exams. There are many methods for solving problems on the extreme values with mathematics knowledge in middle school, such as the definition method, the derivative method, the method of completing the square, the elimination method, the method of combining numbers and shapes as well as the proof of inequalities,etc. Only choose an appropriate method, can a problem be readily solved.

2.1 The definition method

The most important thing in solving relevant problems on the extreme values of a function with the definition method is to master the meaning of the definition and accurately use it. It should be noted that a function must have its value domain but not necessarily have the extreme values.

E.g.1: Suppose the definition domain for $f(x)$ is R, what are correct among the following propositions?

(1) If there is a constant P so that $f(x) \geq P$ is always true for $x \in R$, then P is the minimum of $f(x)$;

(2) If there is $x_0 \in R$ so that $f(x) \geq f(x_0)$ is true for any $x \in R$, then $f(x_0)$ is the minimum of $f(x)$;

(3) If there is $x_0 \in R$ so that $f(x) > f(x_0)$ is true for any $x \in R$ and $x \neq x_0$, then $f(x_0)$ is the minimum of $f(x)$;

Solution: Knowing from the definition of the minimum of a function, (1) is a false proposition: it satisfies the randomness of the definition of the minimum, but so is not for the existence, thus it is false. (2) and (3) are true: in fact, they are equivalent propositions and satisfy the two conditions of the definition of the extreme values.

2.2 The derivative method

E.g.2: Find the extreme values of $f(x) = x^3 + 6x^2 - 15x + 5$ over $[-6,3]$.

Solution: Since $f(x) = x^3 + 6x^2 - 15x + 5$, so $f'(x) = 3x^2 + 12x - 15$,

let $f'(x) = 3x^2 + 12x - 15 = 3(x-1)(x+5) = 0$, we have $x_1 = 1, x_2 = -5$, $f(-6) = 85$, $f(-5) = 105$, $f(1) = -3$, $f(3) = 41$. So, we have $f_{\text{extremely largest}}(-5) = 105$, $f_{\text{extremely smallest}}(1) = -3$. Comparatively, we have $f_{\max}(x) = 105, f_{\min}(x) = -3$. So, the maximum and the minimum of $f(x) = x^3 + 6x^2 - 15x + 5$ over the closed interval $[-6,3]$ are respectively 105 and -3.

2.3 The monotony method

The extreme values of a derivable function over a closed interval come from the function values at two end points of the interval and the extreme values of the function over the interval, while the extreme values also come from the function value at the roots of $f'(x) = 0$. So, solving the extreme values of a derivable function over $[a,b]$ is suggested to be classified into the following five steps.

(1) Find the derivative of the function;

(2) Find x over $[a,b]$ so that $f'(x)=0$, such points are called stagnation points;

(3) Judge the positive or negative of $f'(x)$ at both sides of a stagnation point, and from which to judge the trend of function curve($f'(x)>0$ means ascending and $f'(x)<0$ means descending), the function value ascending at the left of stagnation point and descending at the right is the extremely largest value, otherwise, is the extremely smallest value;

(4) If a function has several stagnation points, then discuss by segments and describe by listing and drawing;

(5) Find the maximum by comparing all extremely largest values with the function values at both end points of the function's definition domain interval and taking the largest one. And so is for taking the minimum.

2.4 The discriminant law

For finding the extreme values of some functions with special forms, transform $f(x)$ appropriately and make it occur as the coefficients of an one-variate quadratic equation having real roots, then use the sufficient and necessary condition $\Delta \geq 0$ that the one-variate quadratic equation has real roots to find the extreme values of $f(x)$.

2.5 The method of completing the square

This method can be used to solve problems when the given function is a quadratic function or can be converted into a quadratic function after transformation.

E.g.3: Find the extreme values of $f(x)=2^{x+2}-3\cdot 4^{x}$ over $[-1,0]$.

Solution: Complete the square, we have $f(x)=2^{x+2}-3\cdot 4^{x}=-3(2^{x}-\frac{2}{3})^{2}+\frac{4}{3}$,

Since $x\in[-1,0]$, so $\frac{1}{2}\leq 2^{x}\leq 1$, thus when $2^{x}=\frac{2}{3}$, i.e. $x=\log_{2}\frac{2}{3}$, $f(x)$ takes the minimum $\frac{4}{3}$;

when $2^{x}=1$, i.e. $x=0$, $f(x)$ takes the minimum 1.

2.5 The elimination method

Of the conditions solving the extreme values of a multivariate function, if some variables can be solved by the multivariate relationships between these conditions, then the elimination method by substitution can be used to convert the multivariate function problem into one-variate function so that the problem gets simplified.

E.g.4: Let $x^2+2y^2=3x$, find the maximum of $u=2x^2+y^2-x$.

Solution: Knowing from the known condition, $y^2=\frac{1}{2}\left(-x^2+3x\right)$ ①

since $-x^2+3x\geq 0$, so $0\leq x\leq 3$, substitute ① into $u=2x^2+y^2-x$ and convert it into an one-variate function, and use the method of completing the square to solve.

2.6 Find the extreme values by the method combining number and shapes

The method combining numbers and shapes is an important problem-solving method whose core is to convert the extreme value problem of a function into a geometric problem for solving by using the function's geometric meaning, which is very intuitive, easy to understand, and has a certain flexibility and works very well for making hard problems simple.

E.g.5: Let $x-y+3=0$, find the extreme values of $S=\sqrt{(x+1)^2+y^2}+\sqrt{(x-1)^2+y^2}$.

Solution: The geometric meaning of this problem lies in finding a point M on $x-y+3=0$ so that the sum of the two distances from M to $(-1,0)$ and $(1,0)$ is the least, as is shown in Fig.3-1.

Suppose the coordinates of A,B are respectively $(-1,0)$ and $(1,0)$, and the equation of the line l is $x-y+3=0$. Knowing from the principle of geometrical optics, when a point light source emits from A and reflected by the mirror surface l to B, then $|AM|+|BM|=|NB|$ is the minimum to be solved.

Suppose the symmetric point of B about l is $N(x_1,y_1)$, then

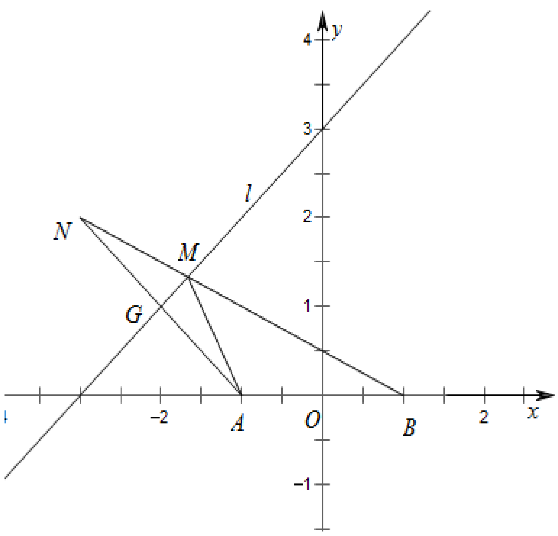

Fig. 3-1

$S_{\min} = |AM| + |BM| = |NB|$, simplify

$$\begin{cases} \dfrac{y_1 - 0}{x_1 + 1} \times 1 = -1 \\ \dfrac{x_1 - 1}{2} - \dfrac{y_1}{2} + 3 = 0 \end{cases}$$

We have $\begin{cases} x_1 + y_1 + 1 = 0 \\ x_1 - y_1 + 5 = 0 \end{cases}$

So, $x_1 = -3, y_1 = 2$

Thus, $S_{\min} = |A \quad| + |B \quad| = |N \quad|$

$$= \sqrt{(-3-1)^2 + (2-0)^2}$$

$$= 2\sqrt{5}$$

2.7 Find extreme values with the U-substitution method

The U-substitution transformation is an important mathematical transformation and has extensive applications in math. The appropriate and flexible application of the U-substitution method can make a complex problem simple.

E.g.6: Let $x^2 + xy + y^2 = 12$. Find the extreme values of $x^2 + y^2$.

Solution: $x = r\cos\theta$, $y = r\sin\theta$ (θ is a parameter), then

$$x^2 + xy + y^2 = r^2(\cos^2\theta + \cos\theta\sin\theta + \sin^2\theta)$$

$$= r^2(1 + \frac{1}{2}\sin 2\theta) = 12.$$

Thus, $x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2 = \dfrac{12}{1 + \frac{1}{2}\sin 2\theta}$.

Since $-1 \le \sin 2\theta \le 1$, when $\sin 2\theta = 1$ (i.e. $x = y = 2$), so $(x^2 + y^2)_{\min} = 8$; when $\sin 2\theta = -1$ (i.e.

$x = y = 2\sqrt{3}$), so $(x^2 + y^2)_{\max} = 24$.

2.8 The proof of an extreme value inequality

Theorem: Let $f(x) = m^x + m^{ax^2+bx+c}$ ($a > 0, m > 1, m \in N$). If a non-negative integer k satisfies

(1) $ak^2 + bk + c + \log_m[-(2ak+b)] - k = 0,$

(2) $ak^2 + bk + c \in \overline{Z^-}$,

then we have

(I) k satisfying the condition (1) is unique;

(II) The minimum of $f(x)$ is $f(x)_{\min} = f(k) = m^k + m^{ak^2+bk+c}$ when $x = k$.

E.g.7: Show that $2^{x+3} + 2^{x^2} \geq 6$, ($x \in R$).

Proof: Let $x + 3 = x'$, then $2^{x+3} + 2^{x^2} = 2^{x'} + 2^{x'^2 - 6x' + 9} = f(x')$, here, $m = 2, a = 1, b = -6, c = 9$.

From the condition (1), we have $k^2 - 6k + 9 + \log_2(6 - 2k) - k = 0$.

Since $k \in \overline{Z^-}$, if $k^2 - 6k + 9 + \log_2(6-2k) - k = 0$ has solutions, then which must satisfy $6 - 2k = 2^p (p \in Z)$,

Knowing from this, the value of k can only be 1 and 2. When verified, only $k = 2$ is the solution of $k^2 - 6k + 9 + \log_2(6-2k) - k = 0$, and $ak^2 + 6k + c = 2^2 - 6 \times 2 + 9 = 1 \in \overline{Z^-}$ satisfies the condition (2), so from the conclusion (II), we have $f(x')_{\min} = f(2) = 2^2 + 2^{2^2 - 6\times 2 + 9} = 6$, then $2^{x+3} + 2^{x^2} \geq 6$ is established.

Notes: Above theorem can be generalized by utilizing relevant knowledge of higher mathematics:

Theorem: Let $f(x) = m^x + m^{ax^2+bx+c}$ ($m > 1, a > 0$). If there is a constant k satisfying $ak^2 + bk + c + \log_m[-(2ak+b)] - k = 0$, then $f(x)_{\min} = f(k) = m^k + m^{ak^2+bk+c}$.

3. Some problems need to be noted in finding the extreme value of a function

3.1 Note the definition domain of a function

We need to note the variation of the definition domain in solving the problems on the extreme values. First, determine the definition domain of the function while seeing the problem. Next, we need to note whether the definition domain is changed as the function transforms in solving the problem, if new variables are introduced, then the valuing ranges of which are also needed to be determined to avoid errors occur in the later problem-solving. Before ending the solving, we need to verify that the independent variables making the function take the extreme values are in the range of the definition domain or not.

E.g.8: Find the extreme values of $y=\frac{\sqrt{1-x}}{x-2}$.

False solution: Square both sides of $y=\frac{\sqrt{1-x}}{x-2}$ simultaneously and clear off its denominator, we have $y^2x^2-(4y^2-1)x+4y^2-1=0$. Since $x\in R$, so $\Delta=(4y^2-1)^2-4y^2(4y^2-1)\ge 0$, simplifying it and we have $4y^2\le 1$. So, $-\frac{1}{2}\le y\le\frac{1}{2}$, thus $y_{\min}=-\frac{1}{2}$, $y_{\max}=\frac{1}{2}$.

Analysis: The errors in this answer lie in squaring both sides of the function expression and clearing off denominator, which enlarges the definition domain of the function.

Correct solution: Square both sides of $y=\frac{\sqrt{1-x}}{x-2}$ simultaneously and clear off its denominator, we have $y^2x^2-(4y^2-1)x+4y^2-1=0$. Since $x\in R$, so $\Delta=(4y^2-1)^2-4y^2(4y^2-1)\ge 0$, simplifying it and we have $4y^2\le 1$. So, $-\frac{1}{2}\le y\le\frac{1}{2}$. Observing that the definition domain of the original function is $x\le 1$, so $x\le 1$, i.e. $x-2<0$, then there must be $y\le 0$. So, $-\frac{1}{2}\le y\le 0$, and thus $y_{\min}=-\frac{1}{2}$, $y_{\max}=0$.

3.2 Note the value domain of a function

Finding the extreme values of a function needs us not only to have the intimidate knowledge of the value domains of several basic elementary functions but also to note the variation of the valuing range of the function while solving the problem.