The inverse trigonometric function

An inverse trigonometric function is a trigonometric function that is non-monotonous over its whole definition domain. So, a trigonometric function has no inverse function over its whole definition domain, but if it is limited to some monotonous interval, then we can discuss the inverse function of a trigonometric function.

I. The inverse sine function

1. Definition: the inverse function of $y = \sin x$ ($x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) is just the inverse sine function, and denoted as $y = \arcsin x$ ($x \in [-1,1]$), which represents that the angle corresponding to the sine function value x over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is $\arcsin x$, that is

$$\boxed{\sin(\arcsin x) = x, \quad x \in [-1,1]}$$

2. The properties of an inverse sine function

(1) Its definition domain is $x \in [-1,1]$, and its value domain is $[-\frac{\pi}{2}, \frac{\pi}{2}]$;

(2) It increases monotonously over its definition domain;

(3) It is an odd-function over $[-1,1]$, that is, $\boxed{\arcsin(-x) = -\arcsin x, \quad x \in [-1,1]}$

(4) The image of $y = \arcsin x$ and the image of $y = \sin x$ ($x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) are symmetric about $y = x$;

(5) The value of $\arcsin(\sin x)$ and the image of $y = \arcsin(\sin x)$: $\boxed{\arcsin(\sin x) = x, \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$

II. The inverse cosine function

By simulating the cases for the inverse sine function, we have

1. Definition: the inverse function of $y = \cos x$ ($x \in [0, \pi]$) is just the inverse cosine function, and denoted as $y = \arccos x$ ($x \in [-1,1]$), which represents that the angle corresponding to the cosine function value x over the interval $[0, \pi]$ is $\arccos x$, that is, $\boxed{\cos(\arccos x) = x, \quad x \in [-1,1]}$.

2. The properties of an inverse cosine function

(1) Its definition domain is $[-1,1]$, and its value domain is $[0,\pi]$;

(2) It decreases monotonously over its definition domain;

(3) It is an either non-odd or non-even function over $[-1,1]$, that is,

$\boxed{\arccos(-x) = -\arccos x,\quad x\in[-1,1]}$

(4) The image of $y=\arccos x$ and the image of $y=\cos x$ ($x\in[0,\pi]$) are symmetric about $y=x$;

(5) The value of arccos(cos$x$) and the image of $y=\arccos(\cos x)$: $\boxed{\arccos(\cos x)=x,\quad x\in[0,\pi]}$

III. The inverse tangent function

1. Definition: the inverse function of $y=\tan x$ ($x\in(-\frac{\pi}{2},\frac{\pi}{2})$) is just the inverse tangent function, and denoted as $y=\arctan x$ ($x\in\mathbf{R}$), which represents that the angle corresponding to the tangent function value x over the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ is $\arctan x$, that is, $\boxed{\tan(\arctan x)=x,\ \ x\in\mathbf{R}}$.

2. The properties of an inverse tangent function

(1) Its definition domain is $\mathbf{R}$, and its value domain is $(-\frac{\pi}{2},\frac{\pi}{2})$;

(2) It increases monotonously over its definition domain;

(3) It is an odd-function over $\mathbf{R}$, that is, $\boxed{\arctan(-x) = -\arctan x,\ \ x\in\mathbf{R}}$

(4) The image of $y=\arctan x$ and the image of $y=\tan x$($x\in(-\frac{\pi}{2},\frac{\pi}{2})$) are symmetric about $y=x$;

⑸ arctan(tan$x$)的值及 $y=\arctan(\tan x)$的图象：(5) The value of arctan(tan$x$) and the image of $y=$ arctan(tan$x$)): $\boxed{\arctan(\tan x)=x,\ \ x\in(-\frac{\pi}{2},\frac{\pi}{2})}$

IV. Typical examples

E.g.1: Show that: (1) $\cos(\arcsin x)=\sqrt{1-x^2}$; $\sin(\arccos x)=\sqrt{1-x^2}$; $\tan(\text{arccot}x)=\frac{1}{x}$.

(2) $\sin(\arctan x) = \dfrac{x}{\sqrt{1+x^2}}$; $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$;

$\cos(\arctan x) = \dfrac{1}{\sqrt{1+x^2}}$; $\tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x}$.

Proof: (1) Let $\arcsin x = \theta$, then $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin\theta = x$, so $\cos\theta = \sqrt{1-x^2}$, that is, $\cos(\arcsin x) = \sqrt{1-x^2}$; and the remaining expressions can be proved similarly.

(2) Let $\arctan x = \alpha$, then $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\tan\alpha = x$. So, $\sec\alpha = \sqrt{1+x^2}$, thus, $\sin\alpha = \tan\alpha\cos\alpha = \dfrac{x}{\sqrt{1+x^2}}$, that is, $\sin(\arctan x) = \dfrac{x}{\sqrt{1+x^2}}$, and the remaining expressions can be proved similarly.

Notes: This problem gives a method calculating inverse trigonometric functions, which views an inverse trigonometric function as an angle in a certain range(that is, the principal value interval of this inverse trigonometric function), and thus transforms the operations of an inverse trigonometric function into the operations of a trigonometric function.

E.g.2: Find a constant c so that $f(x) = \arctan\dfrac{2-2x}{1+4x} + c$ is an odd-function over $(-\frac{1}{4}, \frac{1}{4})$.

Solution: If $f(x)$ is an odd-function over $(-\frac{1}{4}, \frac{1}{4})$, then the necessary condition is $f(0) = 0$, i.e. $c = -\arctan 2$.

When $c = -\arctan 2$, $\tan(\arctan\dfrac{2-2x}{1+4x} - \arctan 2) = \dfrac{\frac{2-2x}{1+4x} - 2}{1 + \frac{2-2x}{1+4x}\cdot 2} = \dfrac{2-2x-2-8x}{1+4x+4-4x} = -2x$.

That is, $f(x) = \arctan(-2x)$; $f(-x) = \arctan(-(-2x)) = \arctan 2x = -f(x)$. So, $f(x)$ is an odd-function over $(-\frac{1}{4}, \frac{1}{4})$.

E.g.3: The valuing range of x satisfying $\arcsin x > \arccos x$ is ( )

$A. \left(0, \dfrac{\sqrt{2}}{2}\right]$ $\qquad B. \left(\dfrac{\sqrt{2}}{2}, 1\right]$

$C.\left[-1,\ \frac{\sqrt{2}}{2}\right)$ $\qquad D.\left[-1,\ 0\right)$

Analysis and solution: This problem involves the unequal relationship, so we need to use the monotony of a function to transform this problem, and separate x from these inverse trigonometric functions due to the valuing range of x, for this purpose, we only need to simultaneously take a certain trigonometric function for $\arcsin x$ and $\arccos x$, lets use the sine function.

If $x \le 0$, then $\arcsin x \in \left[-\frac{\pi}{2},0\right]$, whereas $\arccos x \in \left[\frac{\pi}{2},\ \pi\right]$,

now, $\arcsin x > \arccos x$ is not established, so $x > 0$.

If $x > 0$, then $\arcsin x \in \left(0,\frac{\pi}{2}\right]$, $\arccos x \in \left(0,\frac{\pi}{2}\right]$,

whereas $y = \sin x$ is an increasing function over $\left(0,\frac{\pi}{2}\right]$, and $\arcsin x > \arccos x$, so,

$\sin\left(\arcsin x\right) > \sin\left(\arccos x\right)$, i.e. $x > \sqrt{1-x^2}$, solving this inequality and we have $|x| > \frac{\sqrt{2}}{2}$,

and $0 < x \le 1$, so $\frac{\sqrt{2}}{2} < x \le 1$, and we have (B).

E.g.4: If $0 < \alpha < \frac{\pi}{2}$, then $\arcsin\left[\cos\left(\frac{\pi}{2}+\alpha\right)\right] + \arccos\left[\sin\left(\pi+\alpha\right)\right] = (\ )$

$A.\frac{\pi}{2}$ $\quad B.-\frac{\pi}{2}$ $\quad C.\frac{\pi}{2}-2\alpha$ $\quad D.-\frac{\pi}{2}-2\alpha$

Analysis and solution: This involves the inverse trigonometric operations of trigonometric functions, the method for which is to transform angles to the value domains of corresponding inverse trigonometric functions,

$$\arcsin\left[\cos\left(\frac{\pi}{2}+\alpha\right)\right] = \arcsin(-\sin\alpha) = -\arcsin(\sin\alpha) = -\alpha$$

$$\arccos\left[\sin(\pi+\alpha)\right] = \arccos(-\sin\alpha) = \pi - \arccos(\sin\alpha)$$

$$= \pi - \arccos\left[\cos(\frac{\pi}{2} - \alpha)\right] = \pi - (\frac{\pi}{2} - \alpha) = \frac{\pi}{2} + \alpha,$$

So, the original expression is equal to $(-\alpha) + (\frac{\pi}{2} + \alpha) = \frac{\pi}{2}$, and the answer is （$A$）.

E.g.5: Find: （1） $\sin\left[2\arcsin(-\frac{3}{5})\right]$ （2） $tg(\frac{1}{2}\arccos\frac{1}{3})$

Analysis: $\arcsin\left(-\frac{3}{5}\right)$ represents an angle over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if let $\alpha = \arcsin\left(-\frac{3}{5}\right)$, then we easily have $\sin\alpha = -\frac{3}{5}$, and the original problem is exactly to find the value of $\sin 2\alpha$, then this problem is transformed into a familiar problem solving the value of a trigonometric function, the key for such problems lies in understanding the meaning of a trigonometric expression and its operating order, and using the idea of substituting variables to transform a problem into finding the value of a trigonometric functions.

Solution: (1) Let $\arcsin(-\frac{3}{5}) = \alpha$, then $\sin\alpha = -\frac{3}{5}$

since $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\cos\alpha = \sqrt{1 - \sin^2\alpha} = \frac{4}{5}$

thus, $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \cdot (-\frac{3}{5}) \cdot (\frac{4}{5}) = -\frac{24}{25}$

that is, $\sin\left[2\arcsin(-\frac{3}{5})\right] = -\frac{24}{25}$

(2) Let $\arccos\frac{1}{3} = \alpha$, then $\cos\alpha = \frac{1}{3}$

Since $\alpha \in [0, \pi]$, so $\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{2\sqrt{2}}{3}$

thus, $tg\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha} = \frac{1 - \frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \frac{\sqrt{2}}{2}$

that is, $tg\left[\frac{1}{2}\arccos\frac{1}{3}\right]=\frac{\sqrt{2}}{2}$