Trigonometric equations

E.g.1: Find the valuing range of the real number a so that $\sqrt{a+\sqrt{a+\sin x}}=\sin x$ have real solutions.

Solution: $\sin x\geq 0$, square this expression and we have $\sqrt{a+\sin x}=\sin^2 x-a$, so $a\leq\sin^2 x$,

Squaring and tidying the result, we have $a^2-(2\sin^2 x+1)a+\sin^4 x-\sin x=0$, which is a unary quadratic equation about a.

$=(2\sin^2 x+1)^2-4(\sin^4 x-\sin x)=4\sin^2 x+4\sin x+1=(2\sin x+1)^2$.

So, $a=\frac{1}{2}[2\sin^2 x+1\pm(2\sin x+1)]$, where $a=\sin^2 x+\sin x+1>\sin^2 x$, so discard it.

Thus $a=\sin^2 x-\sin x$, when $0\leq\sin x\leq 1$, we have $a\in[-\frac{1}{4},\ 0]$.

When $a=0$, we have $\sin x=0$ or 1, and the real solutions exist; when $a=-\frac{1}{4}$, $\sin x=\frac{1}{2}$ has real solutions. That is, the valuing of a is $[-\frac{1}{4}, 0]$.

E.g.2: Solve $4\sin^2 x+2\sin x\cos x=1$

Analysis 1: To decrease the power of $\sin^2 x$, we reversely apply the double-angle sine formula to $2\sin x\cos x$, which can be transformed into the structure of the auxiliary angled product, and which further can be transformed into the form of a function expression.

Solution 1: The original equation is transformed to $4\cdot\frac{1-\cos 2x}{2}+\sin 2x=1$, that is, $\sin 2x-2\cos 2x=-1$.

So, $\sqrt{5}\sin(2x-arctg2)=-1$, $\sin(2x-arctg2)=\frac{-\sqrt{5}}{5}$

Thus, $2x-arctg2=2k\pi-\arcsin\frac{\sqrt{5}}{5}$ or $2x-arctg2=2k\pi+\pi+\arcsin\frac{\sqrt{5}}{5}$ $(k\in Z)$

Therefore, $x=k\pi+\frac{1}{2}arctg2-\frac{1}{2}\arcsin\frac{\sqrt{5}}{5}$ or $x=k\pi+\frac{1}{2}arctg2+\frac{\pi}{2}+\frac{1}{2}\arcsin\frac{\sqrt{5}}{5}(k\in Z)$

Analysis 2: Observing that $1=\sin^2 x+\cos^2 x$ and each item of the equation is a quadratic homogeneous expression, we can consider the homogeneous transformation toward tangent function and convert the original equation into an equation with tgx as the unknown number.

Solution 2: The original equation is converted into $3\sin^2 x+2\sin x\cos x-\cos^2 x=0$.

Since $\cos x\neq 0$, simultaneously divide both sides of the equation by $\cos^2 x$, we have

$3tg^2x+2tgx-1=0$, so, $tgx=-1$ or $tgx=\frac{1}{3}$. Thus, $x=k\pi-\frac{\pi}{4}$ or $x=k\pi+arctg\frac{1}{3}$, （$k\in Z$）.

E.g.3: Suppose x_1,x_2 are two roots of $x^2-x\cdot\sin\frac{\pi}{5}+\cos\frac{4\pi}{5}=0$, and $\alpha=arctgx_1$, $\beta=arctgx_2$, find $\alpha+\beta$.

Solution: Since x_1,x_2 are two roots of $x^2-x\cdot\sin\frac{\pi}{5}+\cos\frac{4\pi}{5}=0$, so

$\begin{cases}x_1+x_2=\sin\frac{\pi}{5}>0\\ x_1x_2=\cos\frac{4\pi}{5}<0\end{cases}$, one of x_1,x_2 is positive and the other is negative, and the absolute of the positive root is larger than the absolute of the negative root. So, $0<\alpha+\beta<\frac{\pi}{2}$.

From $tga=x_1$, $tg\beta=x_2$, we have $\begin{cases}tg\alpha+tg\beta=\sin\frac{\pi}{5}\\ tg\alpha\cdot tg\beta=\cos\frac{4\pi}{5}\end{cases}$, so

$tg(\alpha+\beta)=\frac{tg\alpha+tg\beta}{1-tg\alpha\cdot tg\beta}=\frac{\sin\frac{\pi}{5}}{1-\cos\frac{4\pi}{5}}=\frac{\sin\frac{\pi}{5}}{1+\cos\frac{\pi}{5}}=tg\frac{\pi}{10}$, thus $\alpha+\beta=\frac{\pi}{10}$.

E.g.4: Find α so that $\sin 4x\sin 2x-\sin x\sin 3x=\alpha$ has only one solution over $[0,\pi)$.

Analysis and solution:

The original equation can be converted into $-\frac{1}{2}[\cos 6x-\cos 2x]+\frac{1}{2}[\cos 4x-\cos 2x]=\alpha, x\in[0,\pi)$,

$\frac{1}{2}(\cos 4x-\cos 6x)=\alpha\Rightarrow\sin 5x\sin x=\alpha$. Let $f(x)=\sin 5x\sin x$, then

$f\left(\frac{\pi}{2}+x\right)=\sin 5\left(\frac{\pi}{2}+x\right)\cdot\sin\left(\frac{\pi}{2}+x\right)=\sin\left(\frac{\pi}{2}+5x\right)\sin\left(\frac{\pi}{2}+x\right)=\cos 5x\cdot\cos x$,

$f\left(\frac{\pi}{2}-x\right)=\sin 5\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)=\cos 5x\cos x$. That is, $f(x)$ is symmetric about $x=\frac{\pi}{2}$, so

$f(x)=\alpha$ can take the only solution at $x=0$ or $\frac{\pi}{2}$ within $x\in[0,\pi)$.

When $x=0$, $a=0$, but $f(x)=0$ is all established at $x=\frac{\pi}{5},\frac{2}{5}\pi,\frac{3}{5}\pi,\frac{4}{5}\pi$, that is, the solutions are not unique.

When $x=\frac{\pi}{2}$, $a=1$, the solution is unique, which satisfies the requirements in the problem.

To sum up, $a=1$.