Trigonometric identical equations and trigonometric inequalities

I. Basic formulas

1. Some simple trigonometric identical equations

(1) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(2) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(3) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(4) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(5) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(6) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(7) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(8) $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$

We point out here that trigonometric identical equations like (1),(2),(3),(5) and (8) can be used to perform trigonometric substitution, that is, we can perform the substitutions like $x = \tan A$, $y = \tan B$ and $z = \tan C$ under some algebraic problems with given condition like $x + y + z = xyz$, and further use the knowledge on trigonometric functions and the condition $A + B + C = \pi$ to solve the problem.

2. Trigonometric inequalities

(1) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$

(2) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{1}{8}$

(3) $1 + \cos A + \cos B + \cos C < \sin A + \sin B + \sin C$

(4) $\tan A + \tan B + \tan C > \cot A + \cot B + \cot C$

(5) $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$

(6) $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$

II. Typical examples

E.g.1: For ΔABC, show that: $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$.

Proof: Suppose C is an acute angle.

$$\begin{aligned}\cos A+\cos B+\cos C &= \cos A+2\cos\frac{B+C}{2}\cos\frac{B-C}{2}\\ &\le \cos A+2\cos\frac{B+C}{2}=1-2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\\ &=-2(\sin\frac{A}{2}-\frac{1}{2})^2+\frac{3}{2}\le\frac{3}{2}\end{aligned}$$

The equal sign is established when and only when $A=B=C$.

$$\begin{aligned}\cos A+\cos B+\cos C &> \sin B\cos A+\sin A\cos B+\cos C\\ &=\sin(A+B)+\cos C\\ &=\sin C+\cos C\\ &=\sqrt{1+2\sin C\cos C}\\ &\ge 1\end{aligned}$$

E.g.2: For ΔABC, show that: $\cos A\cos B\cos C\le\frac{1}{8}$.

From the conclusion, $\cos A\cos B\cos C\le(\frac{\cos A+\cos B+\cos C}{3})^3\le(\frac{1}{2})^3=\frac{1}{8}$.

E.g.3: For ΔABC, show that: $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$.

Proof: From the conclusion, we have

$$\begin{aligned}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= \frac{\cos A+\cos B+\cos C-1}{4}\\ &\le\frac{\frac{3}{2}-1}{4}=\frac{1}{8}\end{aligned}.$$

E.g.4: Show that: $\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$.

Analysis 1: This problem includes three variables A,B,C, which satisfy $A+B+C=180^\circ$. Fix one first, such as C. Since $A+B=180^\circ-C$, so we perform the sum-and-difference-to-product operation to the left side of the inequality, and converting it into a trigonometric function about $A-B$ for discussing.

Proof 1: First, we suppose C is a constant, then so is $A+B=\pi-C$.

$$\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin C=2\cos\frac{c}{2}\cos\frac{A-B}{2}+\sin C,$$

obviously, for the same C, above expression takes its maximum when $A=B$.

Similarly, we have similar conclusion for the same A or B. So, if any two of A,B,C are different, then $\sin A+\sin B+\sin C$ does not get to its maximum. Therefore, $\sin A+\sin B+\sin C$ gets to its maximum $\frac{3}{2}\sqrt{3}$ when $A=B=C=\frac{\pi}{3}$, that is the original inequality is proved.

Notes: When an inequality includes many variables, we often fix some of them, and find the extreme values of corresponding expression when other variables change, which is called the progressive adjustment method.

Analysis 2: This problem is to prove $\frac{\sin A+\sin B+\sin C}{3}\le\frac{\sqrt{3}}{2}$, Observe the form of the left side, and thus consider the use of the Jensen's inequality for proving.

Proof 2: The function $y=\sin x$ is a convex function over $(0,\pi)$, and thus for any three independent variables $x_1,x_2,x_3\in(0,\pi)$, $\sin(\frac{x_1+x_2+x_3}{3})\ge\frac{\sin x_1+\sin x_2+\sin x_3}{3}$ is always true, and the equal sign is established when $x_1=x_2=x_3$. So, we have $\sin(\frac{A+B+C}{3})\ge\frac{\sin A+\sin B+\sin C}{3}$, therefore, $\frac{\sin A+\sin B+\sin C}{3}\le\sin\frac{180°}{3}=\frac{\sqrt{3}}{2}$, that is, the original inequality is established.

Notes: This method utilizes the properties of a convex function to solve a problem, and trigonometric functions are all convex functions within a certain intervals, so, many trigonometric inequalities can be proved with the properties of a convex function.

E.g.5: Show that: $\tan A+\tan B+\tan C>\cot A+\cot B+\cot C$

Proof: Let $x=\tan A$, $y=\tan B$, $z=\tan C$, then we have $x+y+z=xyz$,

From $(x+y+z)^2>x^2+y^2+z^2\ge xy+xz+yz$, we have

$$x+y+z>\frac{xy+yz+zx}{x+y+z}=\frac{xy+yz+zx}{xyz}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

That is, $\tan A+\tan B+\tan C>\cot A+\cot B+\cot C$

E.g.6: Show that: $x^2+y^2+z^2\ge 2yz\cos A+2xz\cos B+2xy\cos C$

Proof:

$x^2+y^2+z^2-2yz\cos A-2xz\cos B-2xy\cos C=(z-y\cos A-x\cos B)^2+(y\sin A-x\cos B)^2\ge 0$,

so we have $x^2+y^2+z^2\ge 2yz\cos A+2xz\cos B+2xy\cos C$, the equal sign is established when and only when $\dfrac{x}{\sin A}=\dfrac{y}{\sin b}=\dfrac{z}{\sin C}$.

This is the famous embedded inequality, many inequality problems in mathematical contests can be proved with some special A,B,C.

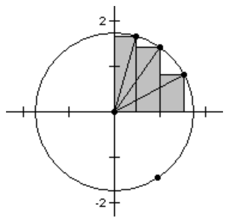
E.g.7: Suppose the real numbers x,y,z satisfy $0<x<y<z<\dfrac{\pi}{2}$, show that:

$\dfrac{\pi}{2}+2\sin x\cos y+2\sin y\cos x>\sin 2x+\sin 2y+\sin 2z$.

Analysis: Convert double-angle sines into single-angle sines and cosines, and associate them with the trigonometric function lines in an unit circle, and further associate the product of any two sines and cosines with the area of a graph.

Proof: This problem is to prove

$\dfrac{\pi}{4}+\sin x\cos y+\sin y\cos z>\sin x\cos x+\sin y\cos y+\sin z\cos z$,



That is, we need to prove $\dfrac{\pi}{4}>\sin x(\cos x-\cos y)+\sin y(\cos y-\cos z)+\sin z\cos z$,

Observing that the right side of above inequality is the sum of the areas of three shadow rectangles within the unit circle shown in the right graph, whereas $\dfrac{\pi}{4}$ is equal to the area of the first quadrantal part of the unit circle, so above inequality is established. To sum up, the original inequality is established.

E.g.8: Suppose the positive real numbers a,b,c satisfy $a+b+c=abc$, show that: $\dfrac{1}{\sqrt{1+a^2}}+\dfrac{1}{\sqrt{1+b^2}}+\dfrac{1}{\sqrt{1+c^2}}\le\dfrac{3}{2}$

Proof: Let $a=\tan A$, $b=\tan B$, $c=\tan C$, where A,B,C are the internal angles of a triangle.

So, the original inequality is equivalent to $\frac{1}{\sqrt{1+\tan^2 A}}+\frac{1}{\sqrt{1+\tan^2 B}}+\frac{1}{\sqrt{1+\tan^2 C}}\le\frac{3}{2}$, that is,

$\cos A+\cos B+\cos C\le\frac{3}{2}$.