Trigonometric identical transformation

I. Principal formulas

1. Formulas on the sum and difference of two angles

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

2. Double-angle and half-angle formulas

(1) Double-angle and half-angle formulas

$$\sin 2\alpha=2\sin\alpha\cos\alpha$$

$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=2\cos^2\alpha-1=1-2\sin^2\alpha$$

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{1-\cos\alpha}{\sin\alpha}$$

(2) Universal formulas

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \qquad \tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

3. Triple-angle formulas

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha\sin(\frac{\pi}{3}+\alpha)\sin(\frac{\pi}{3}-\alpha)$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = 4\cos\alpha\cos(\frac{\pi}{3}+\alpha)\cos(\frac{\pi}{3}-\alpha)$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} = \tan\alpha\tan(\frac{\pi}{3}+\alpha)\tan(\frac{\pi}{3}-\alpha)$$

4. Auxiliary-angle formulas

$a\sin\alpha + b\cos\alpha = \sqrt{(a^2+b^2)}\,\sin(\alpha+\beta)$.

Where, $\sin\beta = \frac{b}{\sqrt{a^2+b^2}}$, $\cos\beta = \frac{a}{\sqrt{a^2+b^2}}$

5. Product-to-sum-and-difference formulas and sum-and-difference-to- product formulas on trigonometric functions

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \qquad \cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

6. Functional values at special angles

$$\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}, \quad \sin 75^\circ = \cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4},$$

$$\tan 15^\circ = \cot 75^\circ = 2-\sqrt{3}, \quad \tan 75^\circ = \cot 15^\circ = 2+\sqrt{3}$$

II. Typical examples

E.g.1: Let $\sin(\alpha-\beta)=\frac{5}{13}$, $\sin(\alpha+\beta)=-\frac{5}{13}$, and $\alpha-\beta\in\left(\frac{\pi}{2},\pi\right)$, $\alpha+\beta\in\left(\frac{3\pi}{2},2\pi\right)$. Find $\sin 2\alpha$ and $\cos 2\beta$.

Solution: since $\alpha-\beta\in\left(\frac{\pi}{2},\pi\right)$, so $\cos(\alpha-\beta)=-\sqrt{1-\sin^2(\alpha-\beta)}=-\frac{12}{13}$.

And since $\alpha+\beta\in\left(\frac{3\pi}{2},2\pi\right)$, so $\cos(\alpha+\beta)=\sqrt{1-\sin^2(\alpha+\beta)}=\frac{12}{13}$.

Thus, $\sin 2\alpha=\sin[(\alpha+\beta)+(\alpha-\beta)]=\sin(\alpha+\beta)\cos(\alpha-\beta)+\cos(\alpha+\beta)\sin(\alpha-\beta)=\frac{120}{169}$,

$\cos 2\beta=\cos[(\alpha+\beta)-(\alpha-\beta)]=\cos(\alpha+\beta)\cos(\alpha-\beta)+\sin(\alpha+\beta)\sin(\alpha-\beta)=-1$.

E.g.2: Find $\tan 20^\circ+4\cos 70^\circ$.

Solution: $\tan 20^\circ+4\cos 70^\circ=\frac{\sin 20^\circ}{\cos 20^\circ}+4\sin 20^\circ$

$$=\frac{\sin 20^\circ+4\sin 20^\circ\cos 20^\circ}{\cos 20^\circ}=\frac{\sin 20^\circ+2\sin 40^\circ}{\cos 20^\circ}$$

$$=\frac{\sin 20^\circ+\sin 40^\circ+\sin 40^\circ}{\cos 20^\circ}=\frac{2\sin 30^\circ\cos 10^\circ+\sin 40^\circ}{\cos 20^\circ}$$

$$=\frac{\sin 80^\circ+\sin 40^\circ}{\cos 20^\circ}=\frac{2\sin 60^\circ\cos 20^\circ}{\cos 20^\circ}=\sqrt{3}.$$

E.g.3: Suppose that α is an acute angle and β is an obtuse angle, and

$\sec(\alpha-2\beta),\sec\alpha,\sec(\alpha+2\beta)$ form a arithmetic progression, find $\dfrac{\cos\alpha}{\cos\beta}$.

Analysis: Transform secant into cosine and depress powers, and transform operation method.

Solution: Transform secant into cosine by the conditions, we have

$$\frac{2}{\cos\alpha}=\frac{1}{\cos(\alpha-2\beta)}+\frac{1}{\cos(\alpha+2\beta)},$$

$$\frac{2}{\cos\alpha}=\frac{\cos(\alpha+2\beta)+\cos(\alpha-2\beta)}{\cos(\alpha+2\beta)\cos(\alpha-2\beta)},$$

$$\frac{2}{\cos\alpha}=\frac{2\cos\alpha\cos 2\beta}{\frac{1}{2}(\cos 2\alpha+\cos 4\beta)},\quad \cos 2\alpha+\cos 4\beta=2\cos^2\alpha\cos 2\beta,$$

$$\cos 2\alpha+\cos 4\beta=(1+\cos 2\alpha)\cos 2\beta\ \cos 2\alpha(1-\cos 2\beta)=\cos 2\beta-\cos 4\beta,$$

$$\cos 2\alpha(1-\cos 2\beta)=\cos 2\beta-2\cos^2 2\beta+1$$

$\cos 2\alpha(1-\cos 2\beta)=(1-\cos 2\beta)(2\cos 2\beta+1)$, that is, $\cos 2\alpha=2\cos 2\beta+1$, $\cos^2\alpha=2\cos^2\beta$

Since α is an acute angle, β is an obtuse angle, we have $\dfrac{\cos\alpha}{\cos\beta}=-\sqrt{2}$.

E.g.4: Let $0<\alpha,\beta<\dfrac{\pi}{2}$, show that: $\alpha+\beta=\dfrac{\pi}{2}$ is the sufficient and necessary condition for $\sin^2\alpha+\sin^2\beta=\sin^2(\alpha+\beta)$ to be established.

Analysis: Use relevant formulas to expand this expression directly.

Solution 1: The sufficiency is obvious, we are to prove the necessity.

From $\sin^2\alpha+\sin^2\beta=\sin^2(\alpha+\beta)$, we have

$\sin^2\alpha+\sin^2\beta=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2$

That is, $\sin^2\alpha+\sin^2\beta=\sin^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+2\sin\alpha\cos\alpha\sin\beta\cos\beta$, simplifying it and we have $2\sin^2\alpha\sin^2\beta=2\sin\alpha\sin\beta\cos\alpha\cos\beta$, that is, $\sin\alpha\sin\beta\cos(\alpha+\beta)=0$, from $0<\alpha,\beta<\frac{\pi}{2}$, we have $\alpha+\beta=\frac{\pi}{2}$.

Solution 2: Solve it by constructing a triangle. Construct $\Delta ABC,\angle A=\alpha,\angle B=\beta$, then

$\angle C=\pi-(\alpha+\beta)$, since $\sin^2\alpha+\sin^2\beta=\sin^2(\alpha+\beta)$, that is, $\sin^2\alpha+\sin^2\beta=\sin^2 C$, so $a^2+b^2=c^2$, thus we have $C=\frac{\pi}{2}$, i.e. $A+B=\frac{\pi}{2}$.

E.g.5: Find $\cos^2 10+\cos^2 50-\sin 40\sin 80$.

Analysis: The basic methods solving this problem are power-depressing, sum-and-difference-to-product and construction of structural features.

Solution 1: Observe that $\sin 40^0=\cos 50^0,\sin 80^0=\cos 10^0$, and the trigonometric expression is symmetric about $\cos 10^0,\cos 50^0$, so we can construct a binary symmetric substitution for finding this value. Let $\cos 10^0=a+b,\cos 50^0=a-b$, then $a=\frac{1}{2}(\cos 10^0+\cos 50^0)=\frac{\sqrt{3}}{2}\cos 20^0$,

$b=\frac{1}{2}(\cos 10^0-\cos 50^0)=\frac{1}{2}\sin 20^0$, so

The original expression $=\cos^2 10+\cos^2 50-\cos 50^0\cos 10^0$

$=(a+b)^2+(a-b)^2-(a-b)(a+b)=a^2+3b^2$

$=(\frac{\sqrt{3}}{2}\cos 20^0)^2+3(\frac{1}{2}\sin 20^0)^2=\frac{3}{4}$.

Solution 2: use $\cos^2 10^0 + \sin^2 10^0 = 1, \cos^2 50^0 + \sin^2 50^0 = 1$ to construct a dual model for solving.

Let $A = \cos^2 10 + \cos^2 50 - \sin 40 \sin 80$, $B = \sin^2 10 + \sin^2 50 - \cos 40 \cos 80$, then $A + B = 2 - \cos 40^0$, $A - B = \cos 20^0 + \cos 100^0 + \cos 120^0 = \cos 40^0 - \frac{1}{2}$, then we have $A = \frac{3}{4}$.

Notes: Analyzing the structural features of a trigonometric expression plays very important role in solving a problem, which tends to reveal the nature of a problem. Surely, this problem can also be solved by other methods like constructing a triangle.

E.g.6: Find $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$.

Analysis: solve from two perspectives, basic methods and construction methods.

Solution 1: (reverse using of sum-and-difference-to-product formulas)

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} == 2\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -2\cos\frac{\pi}{5}\cos\frac{2\pi}{5},$$

Multiple simultaneously the numerator and denominator by $4\sin\frac{\pi}{5}$, reversely use double-angle formulas for consecutive two times, we can have the value $-\frac{1}{2}$.

Solution 2: (construct a dual expression for solving)

Let $x = \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}, y = \cos\frac{2\pi}{5} - \cos\frac{4\pi}{5}$,

$$xy = \cos^2\frac{2\pi}{5} - \cos^2\frac{4\pi}{5} = \frac{1}{2}(1 + \cos\frac{4\pi}{5}) - \frac{1}{2}(1 + \cos\frac{8\pi}{5})$$

$= \frac{1}{2}(\cos\frac{4\pi}{5} - \cos\frac{2\pi}{5}) = -\frac{1}{2}y$. Eliminate $y(y > 0)$, we have $x = -\frac{1}{2}$.

Solution 3: (construct an equation by self-substitution for solving)

$x = \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = \cos\frac{2\pi}{5} - \cos\frac{\pi}{5} < 0$, square both sides of this equation, we have $x^2 = \cos^2\frac{2\pi}{5} + \cos^2\frac{4\pi}{5} + 2\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}$

$$=\frac{1}{2}(1+\cos\frac{4\pi}{5})+\frac{1}{2}(1+\cos\frac{8\pi}{5})+2\frac{\sin\frac{4\pi}{5}\sin\frac{8\pi}{5}}{2\sin\frac{2\pi}{5}\cdot 2\sin\frac{4\pi}{5}}$$

$=\frac{1}{2}+\frac{1}{2}(\cos\frac{4\pi}{5}+\cos\frac{8\pi}{5})$ $=\frac{1}{2}+\frac{1}{2}(\cos\frac{4\pi}{5}+\cos\frac{2\pi}{5})$ $=\frac{1}{2}+\frac{x}{2}$, so we have $x^2=\frac{1}{2}+\frac{1}{2}x$, thus, $x=-\frac{1}{2}$.

Solution 4: (construct isomorphic equation)

Let $\cos x+\cos 2x=\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}$, then $\cos x=\cos\frac{2\pi}{5},\cos\frac{4\pi}{5}$ simultaneously satisfy the isomorphic equation. From double-angle formulas, we have the quadratic equation

$2\cos^2 x+\cos x-(1+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5})=0$, which shows that $\cos\frac{2\pi}{5},\cos\frac{4\pi}{5}$ are two roots for $2y^2+y-(1+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5})=0$, and are all roots. From the relationship between roots and coefficients, we have $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$.

E.g.7: Use of triple-angle formulas

(1) Suppose x is an acute angle and satisfies $\frac{\cos 3x}{\cos x}=\frac{1}{3}$, find $\frac{\sin 3x}{\sin x}$.

Solution: from triple-angle formulas, we have $\frac{\cos 3x}{\cos x}=\frac{4\cos^3 x-3\cos x}{\cos x}=4\cos^2 x-3$ and $\frac{\sin 3x}{\sin x}=\frac{3\sin x-4\sin^3 x}{\sin x}=3-4\sin^2 x$, so $\frac{\sin 3x}{\sin x}-\frac{\cos 3x}{\cos x}=2$, thus $\frac{\sin 3x}{\sin x}=\frac{7}{3}$.

(2) Show that: $\sin 3\theta=4\sin\theta\sin(\frac{\pi}{3}+\theta)\sin(\frac{\pi}{3}-\theta)$.

Proof:

证明：$\sin 3\theta=3\sin\theta-4\sin^3\theta$

$$=4\sin\theta(\frac{3}{4}-\sin^2\theta)$$
$$=4\sin\theta(\frac{3}{4}\cos^2\theta-\frac{1}{4}\sin^2\theta)$$
$$=4\sin\theta[(\frac{\sqrt{3}}{2}\cos\theta)^2-(\frac{1}{2}\sin\theta)^2]$$
$$=4\sin\theta(\sin 60^\circ\cos\theta+\cos 60^\circ\sin\theta)(\sin 60^\circ\cos\theta-\cos 60^\circ\sin\theta)$$
$$=4\sin\theta\sin(60^\circ+\theta)\sin(60^\circ-\theta)$$

(3) Show that: $\tan 3\alpha = \tan\alpha \tan(\frac{\pi}{3}+\alpha)\tan(\frac{\pi}{3}-\alpha)$

Proof:

$$\tan\alpha \tan(\frac{\pi}{3}+\alpha)\tan(\frac{\pi}{3}-\alpha) = \tan\alpha \frac{\sqrt{3}+\tan\alpha}{1-\sqrt{3}\tan\alpha}\frac{\sqrt{3}-\tan\alpha}{1+\sqrt{3}\tan\alpha} = \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}$$

$$\tan 3\alpha = \frac{\tan\alpha+\tan 2\alpha}{1-\tan\alpha\tan 2\alpha} = \frac{\tan\alpha+\frac{2\tan\alpha}{1-\tan^2\alpha}}{1-\tan\alpha\frac{2\tan\alpha}{1-\tan^2\alpha}} = \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}$$

$$\tan 3\alpha = \tan\alpha \tan(\frac{\pi}{3}+\alpha)\tan(\frac{\pi}{3}-\alpha)$$