LTE Lemma (Lift The Exponent)

Gary Gao

March 25, 2020

1 Problem 1

$v_5(7^{120} - 2^{120}) = v_5(7-2) + v_5(120) = 2$

2 Problem 2

Let $t = 149^n - 2^n$. Obviously the minimal n occurs when $v_3(t) = 3$, $v_5(t) = 5$, and $v_7(t) = 7$.
Notice that $149 - 2 = 147 = 3 \cdot 7^2$. So we can apply LTE directly for $v_3(t)$ and $v_5(t)$.

$$3 = v_3(t) = v_3(147) + v_3(n) = 1 + v_3(n)$$

$$7 = v_7(t) = v_7(147) + v_7(n) = 2 + v_7(n)$$

Therefore, $v_3(n) = 2$, $v_7(n) = 5$. Now consider $v_5(n)$. Notice that $149 \equiv -1 \mod 5$, we can discover that $149^4 \equiv 2^4 \equiv 1 \mod 5$. So we must have n divisible by 4 (since the squares and cubes do not satisfy this condition). Furthermore, $149^4 \equiv 1 \mod 25$ while $2^4 \equiv 16 \mod 25$. So $v_5(149^4 - 2^4) = 1$.
Apply LTE,

$$v_5(149^k - 2^k) = v_5(149^4 - 2^4) + v_5(4k) = 1 + v_5(k)$$

Therefore, $n = 4k$ is divisible by $4 \cdot 5^4$. So the minimal n must be divisible by $2^2 \cdot 3^2 \cdot 5^4 \cdot 7^5$, which means it would be the minimal n. It has $(2+1)(2+1)(4+1)(5+1) = 270$ divisors.

3 Problem 3

First notice that n must be odd. Obviously $n = 1$ works, then consider when $n > 1$
Suppose that the least prime factor of n is q. Therefore, since $n^2 \mid (2^n + 1)$, we know that $2^n \equiv -1 \mod q$. So $2^{2n} \equiv 1 \mod q$. Also using Euler's totient function (or Fermat's little theorem), we know that $2^{q-1} \equiv 1 \mod q$.
So the following must be true: $\text{ord}_q 2 \mid \gcd(q-1, 2n)$. Since n is divisible by q,$\gcd(q-1, 2n) = 2$. Since $\text{ord}_q 2$ cannot be 1, it must be 2. So $q \mid (2^2 - 1) \Rightarrow q = 3$
Then consider $v_3(n)$. Because $n^2 \mid 2^n + 1$,

$$v_3(2^n + 1) \geq 2v_3(n)$$

$$v_3(n) + 1 \geq 2v_3(n)$$

$$v_3(n) \leq 1$$

Obviously $v_3(n)$ cannot be 0 since 3 is the least prime factor of n. So $v_3(n) = 1$. Then we check $n = 3$ and it works. Then let $n = 3k$ where $k \geq 5$ is an odd number not divisible by 3.
So

$$9k^2 \mid (2^{3k} + 1)$$

$$\iff k^2 \mid 8^k + 1$$

We can use the same argument as before to find the least prime factor of k, which is 7.
However, for any $8^k + 1$, $8^k + 1 \equiv 2 \mod 7$, which gives us a contradiction. Therefore, the only possible values for n are 1 and 3.

4 Problem 4

Let $d = \text{ord}_{3^k} 2$. Since $2^d \equiv 1 \mod 3$, obviously d is even.
So let $d = 2n$. Then $3^k \mid (4^n - 1)$. By LTE,

$$k \leq v_3(4^n - 1) = v_3(3) + v_3(n) = 1 + v_3(n)$$

$$\Rightarrow v_3(n) \geq k - 1$$

The smallest possible n is 3^{k-1}. So the smallest possible d is $2 \cdot 3^{k-1} = \phi(3^k)$, which indicates that 2 is a primitive root of 3^k.